Master's Thesis

Boundedness of The Bilinear Calderón-Zygmund Operator and L^p- Estimation Tools and Techniques

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May 24, 2018

HELSINGIN YLIOPISTO — HELSINGFORS UNIVERSITET — UNIVERSITY OF HELSINKI

Tiedekunta/Osasto — Fakultet/Sektion — Faculty	Laitos — Institution — Department
Matemaattis-luonnontieteellinen	Matematiikan ja tilastotieteen laitos

Tekijä — Författare — Author

Tuomas Oikari

Työn nimi — Arbetets titel — Title

Boundedness of The Bilinear Calderón-Zygmund Operator and
L^p- Estimation Tools and Techniques

Oppiaine — Läroämne — Subject

Matematiikka

Työn laji — Arbetets art — Level	Aika — Datum — Month and year	Sivumäärä — Sidoantal — Number of pages
Pro gradu -tutkielma	Toukokuu 2018	58 s.

Tiivistelmä — Referat — Abstract

Tämän työn päämäärä on kehitellä L^p-estimointi tekniikoita, soveltaa niitä mallioperaattoreiden tutkimiseen, ja edelleen, siirtää näiden mallioperaattoreiden rajoittuneisuus ominaisuuksia bilineaariselle Calderón-Zygmund -operaattorille. Tämä siirtäminen tehdään olettamalla tunnetuksi Bilineaarisen Calderón-Zygmund -operaattorin esityslause satunnaistettujen mallioperaattoreiden summana.

Oletamme tunnetuksi hieman interpolointi ja maksimaalifunktioiden teoriaa.

Aloitamme esittelemällä hyödyllisiä peruskäsitteitä, mm. neliöfunktion, sen kautta L^p -avaruuksien karakterisaation, ja Haarin kannan.

Kappaleissa kaksi ja kolme määrittelemme mallioperattorit: shiftit ja paratulot ja todistamme niitä koskevia vahvoja estimaatteja L^p -avaruuksissa, kun $p > 1$. Tarkoituksemme on interpoloida vahvoja estimaatteja quasi-Banach alueelle, siis kun $p < 1$, ja tätä varten todistamme heikkoja päätepiste estimaatteja.

Neljäs kappale on omistettu interpolointitekniikan esittelemiselle, joka mahdollistaa edellämainitun interpoloinnin.

Viidennessä ja viimeisessä kappaleessa määrittelemme bilineaarisen Calderón-Zygmund operaattorin, kokoamme yhteen aikaisemmin kehitetyn teorian tuloksia ja vedämme näistä tuloksista lyhyehkönä korollaarina bilineaarisen Calderón-Zygmund -operaattorin rajoittuneisuuden.

Avainsanat — Nyckelord — Keywords

Harmoninen analyysi, estimointi, Bilineaariset Calderón-Zygmund operaattorit

Säilytyspaikka — Förvaringsställe — Where deposited

Kumpulan tiedekirjasto

Muita tietoja — Övriga uppgifter — Additional information

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Introduction

Assuming the representation theorem given in [3] for bilinear Calderón-Zygmund operators via what we call model operators, the goal of this master's thesis is to develop all-around-useful tools, apply them in the study of model operators, and then with the results, conclude the boundedness of the bilinear Calderón-Zygmund operator as a mapping $L^p \times L^q \longrightarrow L^r$ for exponents $1 < p, q \leq \infty$ and $r > 1/2$ that satisfy the relation $1/p + 1/q = 1/r$.

We begin by familiarizing ourselves with the dyadic system, show how to represent functions as martingale differences, and define the square function. Then we prove a characterization of L^p, for $p > 1$, via the square function which reveals its usefulness. We conclude chapter one by proving an equality that links martingale differences to Haar functions. This equality combined with the martingale difference representation allows us to easily obtain representation of functions in the Haar-basis.

In chapters two and three we introduce two classes of operators, namely, bilinear shifts and paraproducts, which are what we call model operators. First we prove strong estimates in the Banach range for bilinear shifts. Then, in order to tackle bilinear paraproducts in the Banach range, we move to study linear paraproducts. To begin, we expand our toolkit by introducing an interesting estimate concerning sparse families. Then, for the first time, we employ the full strength of our so-far developed tools, proving a direct L^p-estimate without falling to case study. Chapter two is concluded with strong estimates in the Banach range for bilinear paraproducts.

To move into the quasi-Banach range, we introduce a dualization lemma that gives a characterization of boundedness of an operator mapping to the weak L^r-space, for $r > 0$. Using this, we then proceed to prove weak end-point estimates for both bilinear shifts and paraproducts. The quasi-Banach estimates could in our setting be proved by an approach employing the Calderón-Zygmund decomposition. The drawback of this method would be that it does not generalize to the bi-parameter setting, thus it is avoided.

In chapter four, we introduce a method of interpolation that allows us to conclude strong L^p-estimates for bilinear shifts and paraproducts in the quasi-Banach range from the estimates proved in chapters two and three.

In the last chapter, we define the bilinear Calderón-Zygmund operator and show that by assuming the bilinear Calderón-Zygmund operator representation theorem, we can, by the tools developed in the previous chapters, prove the fore-set boundedness of Calderón-Zygmund operators.

Acknowledgements

I thank academy research fellow Henri Martikainen for the thorough explanation of the topics found here. I also thank Emil Vuorinen for teaching of the topic of multilinear interpolation contained in chapter four.

1 Basic results and square function estimates

Every single operator to be faced in Chapters 1 and 2 will have as a basis of its definition some system of dyadic cubes; everything that follows will have some dyadic system at its core.

1.1 The dyadic system and preliminary definitions

We begin by defining some core concepts and noting the fundamental basic properties of dyadic systems.

Definition 1.2. *If $Q \subset \mathbb{R}^n$ is a cube, its side-length is denoted by $l(Q)$. A collection of cubes $\mathcal{D}$ is called a dyadic grid if*

$$\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k,$$

where $\mathcal{D}_k = \{Q \in \mathcal{D} : l(Q) = 2^{-k}\}$ is a disjoint collection of cubes in $\mathbb{R}^n$ such that

$$\mathbb{R}^n = \bigcup_{Q \in \mathcal{D}_k} Q,$$

and if $Q \in \mathcal{D}_k$, then

$$Q = \bigcup_{\substack{R \in \mathcal{D}_{k+1} \\ R \subset Q}} R.$$

Assume that $\mathcal{F}$ is some sub-collection of a dyadic grid $\mathcal{D}$. The first generation of children of a cube Q with respect to the collection $\mathcal{F}$ is defined as

$$ch_{\mathcal{F}}(Q) = \Big\{ R \in \mathcal{F} : R \subsetneq Q, \text{ and if exists } S \in \mathcal{F} \text{ such that } R \subset S \subsetneq Q, \text{ then } S = R \Big\}.$$

We iterate this to acquire children of the n:th generation:

$$ch^{n+1}_{\mathcal{F}}(Q) = \Big\{ R \in \mathcal{F} : \text{ for some } S \in ch^n_{\mathcal{F}}(Q),\ R \in ch_{\mathcal{F}}(S) \Big\}.$$

If $\mathcal{F} = \mathcal{D}$, then a shorthand $R^{(n)} = Q$, or even $R^n = Q$ if there is no possibility of confusion, will stand for $R \in ch^n_{\mathcal{F}}(Q)$.

Remark 1.3. A standard example of a dyadic grid, or system, on $\mathbb{R}^n$ is given by

$$\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k, \qquad \mathcal{D}_k = \Big\{ 2^{-k}\Big([0,1)^n + x\Big) : x \in \mathbb{Z}^n \Big\}.$$

The properties of dyadic systems gathered together into the following remark will be referred to as the "basic properties of dyadic grids and cubes".

Remark 1.4. Let $\mathcal{D}$ be a dyadic grid and $\mathcal{F} \subset \mathcal{D}$. Then:

1. If $Q, R \in \mathcal{F}$, then $Q \cap R \in \{Q, R, \emptyset\}$.

2. For $Q \in \mathcal{F}$ and $n \in \mathbb{N}$, the collection $ch^n_{\mathcal{F}}(Q)$ is disjoint.

3. For each $R \in \mathcal{F}$ and $n \in \mathbb{N}$, $\Pi^n_{\mathcal{F}}(R)$ is unique, on the condition that it exists.

Definition 1.5. *We write $A \lesssim B$ if there is an absolute constant $C > 0$ (depending only on some fixed constants like n, p, q etc.) so that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. If the constant C has dependence on some parameters, say $n, p, q, \dots$ then one may indicate this by writing $C_{n,p,q,\dots}$, $\lesssim_{n,p,q,\dots}$ or $\sim_{n,p,q,\dots}$.*

The measure to be used is the Lebesgue measure. Functions f considered will be measurable and generally map $f : \mathbb{R}^n \longrightarrow \mathbb{C}$. The set of all such measurable functions is denoted by $L^0(\mathbb{R}^n)$.

Definition 1.6. *Let $p > 0$ and f be a measurable function and define*

$$\|f\|_{L^p} = \Big(\int_{\mathbb{R}^n} |f|^p \Big)^{1/p}.$$

Let $1 \leq p < \infty$. The space of p-integrable functions $L^p(\mathbb{R}^n)$ is

$$L^p(\mathbb{R}^n) = \{f : \mathbb{R}^n \longrightarrow \mathbb{C} : f \text{ is measurable and } \|f\|_{L^p} < \infty\}.$$

Let $p = \infty$. For a measurable function f define

$$\|f\|_{L^\infty} = \operatorname{esssup}(f) = \inf\{\alpha : |\{x : |f(x)| > \alpha\}| = 0\}.$$

The space of essentially bounded functions is

$$L^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \longrightarrow \mathbb{C} : f \text{ is measurable and } \|f\|_{L^\infty} < \infty\}.$$

A shorthand L^p will stand for $L^p(\mathbb{R}^n)$.

Definition 1.7. *A function f is locally integrable, $f \in L^1_{loc}$, if for all compact sets K, $f 1_K \in L^1$.*

1.8 Martingale representation

For the rest of the chapter we fix a dyadic grid $\mathcal{D}$.

If a function is p-integrable, it can be represented as a martingale according to the following theorem.

Theorem 1.9. *Let $1 \leq p < \infty$, $f \in L^p$ and define the function ψ as*

$$\psi = \sum_{Q \in \mathcal{D}} \Delta_Q f,$$

where

$$\Delta_Q f = \sum_{R \in ch(Q)} (\langle f \rangle_R - \langle f \rangle_Q) 1_R,$$

where

$$\langle f \rangle_Q = \frac{1}{|Q|} \int_Q f.$$

Then $f = \psi$ a.e.[1] *and ψ is well-defined.*

Proof. Assume that $x \in \mathbb{R}^n$. By $Q_m(x)$ we denote the cube of side-length 2^{-m} that contains the point x. Denote

$$\psi_m^n = \sum_{\substack{Q \in \mathcal{D} \\ 2^m < l(Q) \leq 2^n}} \Delta_Q f$$

and assume that $m < n$. Then

$$\begin{aligned} \psi_m^n(x) = \sum_{\substack{Q \in \mathcal{D} \\ 2^m < l(Q) \leq 2^n}} \Delta_Q f(x) &= \sum_{\substack{x \in Q \in \mathcal{D} \\ 2^m < l(Q) \leq 2^n}} \sum_{x \in R \in ch(Q)} (\langle f \rangle_R - \langle f \rangle_Q) 1_R(x) \\ &= \langle f \rangle_{Q_{-m}(x)} - \langle f \rangle_{Q_{-n}(x)}. \end{aligned}$$

As $f \in L^p$

$$\lim_{n \to -\infty} \langle f \rangle_{Q_{-n}(x)} = 0,$$

and by the Lebesgue differentiation theorem

$$\lim_{m \to \infty} \langle f \rangle_{Q_{-m}(x)} = f(x) \qquad \text{a.e.}$$

Thus

$$\psi(x) := \lim_{\substack{n \to -\infty \\ m \to \infty}} \psi_m^n(x) = f(x) \qquad \text{a.e.}$$

□

Definition 1.10. *Let $f \in L^1_{loc}$. The dyadic maximal operator $\mathcal{M}_{\mathcal{D}}$ associates to the function f the function*

$$\mathcal{M}_{\mathcal{D}} f := \sup_{Q \in \mathcal{D}} 1_Q \langle |f| \rangle_Q.$$

Definition 1.11. *Let $f \in L^1_{loc}$. The Hardy-Littlewood maximal operator $\mathcal{M}$ associates to the function f the function*

$$\mathcal{M} f := \sup_{Q} 1_Q \langle |f| \rangle_Q,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$.

[1] almost everywhere

We recall the well-known Hardy-Littlewood inequality (see e.g. [5]).

Lemma 1.12. *If* $1 < p < \infty$ *and* $f \in L^p$, *then*

$$\|\mathcal{M}_{\mathcal{D}} f(x)\|_{L^p} \lesssim \|\mathcal{M} f(x)\|_{L^p} \lesssim \|f\|_{L^p}.$$

We also have the following representation

Theorem 1.13. *If* $1 < p < \infty$ *and* $f \in L^p$ *then*

$$\lim_{\substack{n \to \infty \\ m \to -\infty}} \|\psi_m^n - f\|_{L^p} = 0,$$

where ψ_m^n *is as in the proof of theorem 1.9.*

Proof. Reusing part of the previous calculation shows that

$$|\psi_m^n(x)| = |\langle f \rangle_{Q_{-m}(x)} - \langle f \rangle_{Q_{-n}(x)}| \leq 2\mathcal{M}_{\mathcal{D}} f(x).$$

By lemma 1.12 $\mathcal{M}_{\mathcal{D}} f \in L^p$. As $\psi_m^n \longrightarrow f$ point-wise a.e. by theorem 1.9, the claim then follows by the dominated convergence theorem. □

1.14 Square function

Next we define the dadic square function and go through some of its basic properties.

Definition 1.15. *Let* $f \in L^1_{loc}$. *The dyadic square function* $S_{\mathcal{D}}$ *associates to the function* f *the function*

$$S_{\mathcal{D}} f = (\sum_{Q \in \mathcal{D}} |\Delta_Q f|^2)^{\frac{1}{2}}.$$

We use the words dyadic square operator and dyadic square function interchangeably and write them shortly as square operator and square function.

For the next lemma we note that

$$\int \Delta_Q f = \int \sum_{R \in ch(Q)} \Big(\langle f \rangle_R - \langle f \rangle_Q\Big) 1_R = \sum_{R \in ch(Q)} \int_R f - \sum_{R \in ch(Q)} \frac{|R|}{|Q|} \int_Q f = 0.$$

Lemma 1.16. *The operators* Δ_Q, $Q \in \mathcal{D}$ *are "orthogonal", i.e. if* $Q \neq R$, *then for functions* $f, g \in L^1_{loc}$

$$\int \Delta_Q f \Delta_R g = 0.$$

Proof. Assume that $f, g \in L^1_{loc}(\mathbb{R}^n)$. Let $Q, R \in \mathcal{D}$. If $Q \cap R = \emptyset$, then

$$\int_{\mathbb{R}^n} \Delta_R g \Delta_Q f = \int_{\emptyset} \Delta_R g \Delta_Q f = 0.$$

Assuming $Q \cap R \neq \emptyset$, by basic properties of the dyadic systems either $R \subset Q$ or $Q \subset R$. Assume that $R \subset Q$ and $R \neq Q$. Then since $\Delta_R f$ is supported on R and $\Delta_Q f$ is constant on R we have that

$$\int \Delta_R g \Delta_Q f = \langle \Delta_Q f \rangle_R \int \Delta_R g = 0.$$

□

Theorem 1.17. *If $f \in L^2$, then $\|S_{\mathcal{D}} f\|_{L^2} = \|f\|_{L^2}$.*

Proof. By lemma 1.16 and theorem 1.13

$$\|S_{\mathcal{D}} f\|_{L^2}^2 = \int \sum_{Q \in \mathcal{D}} |\Delta_Q f|^2 = \sum_{Q \in \mathcal{D}} \|\Delta_Q f\|_{L^2}^2 \overset{1.16}{=} \|\sum_{Q \in \mathcal{D}} \Delta_Q f\|_{L^2}^2 \overset{1.13}{=} \|f\|_{L^2}^2.$$

□

The following fundamental lemma is known as the Calderón-Zygmund decomposition.

Lemma 1.18. *Let $f \in L^1$ and $\lambda > 0$. Then there exists a disjoint collection of dyadic cubes Ω_λ and functions g and b_Q, where $Q \in \Omega_\lambda$, such that*

1. *$f = g + \sum_{Q \in \Omega_\lambda} b_Q$, and we define $b = \sum_{Q \in \Omega_\lambda} b_Q$,*
2. *$\|g\|_{L^\infty} \leq 2^n \lambda$ and $\|g\|_{L^1} \leq \|f\|_{L^1}$,*
3. *$\int b_Q = 0$,*
4. *$|\bigcup_{Q \in \Omega_\lambda}| \leq \|f\|_{L^1}/\lambda$.*

Proof. We define Ω_λ to be the collection of maximal dyadic cubes $Q \in \mathcal{D}$ that satisfy

$$\langle |f| \rangle_Q > \lambda.$$

As $f \in L^1$, the maximal cubes exist and the collection Ω_λ is well-defined. Since the cubes in Ω_λ are maximal, they are disjoint. By disjointness and maximality of the cubes in Ω_λ

$$|\bigcup_{Q \in \Omega_\lambda} Q| = \sum_{Q \in \Omega_\lambda} |Q| \leq \sum_{Q \in \Omega_\lambda} \frac{1}{\lambda} \int_Q |f| \leq \frac{\|f\|_{L^1}}{\lambda}.$$

Let's define

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R}^n \setminus \bigcup \Omega_\lambda \\ \langle f \rangle_Q & \text{if } x \in Q \in \Omega_\lambda, \end{cases}$$

and
$$b = f - g = \sum_{Q \in \Omega_\lambda} (f - \langle f \rangle_Q) 1_Q = \sum_{Q \in \Omega_\lambda} b_Q.$$
Then
$$\|g\|_{L^1} = \int_{\mathbb{R}^n \setminus \cup \Omega_\lambda} |f| + \sum_{Q \in \Omega_\lambda} \int_Q |\langle f \rangle_Q| \leq \int_{\mathbb{R}^n \setminus \cup \Omega_\lambda} |f| + \sum_{Q \in \Omega_\lambda} \int_Q \langle |f| \rangle_Q = \|f\|_{L1}.$$
and
$$\int_Q b_Q = \int_Q f - \int_Q f = 0.$$
Assume that $x \in Q \in \Omega_\lambda$. Then by maximality of the cube Q
$$|g(x)| = |\langle f \rangle_Q| \leq \langle |f| \rangle_Q = \frac{1}{|Q|} \int_Q |f| \leq \frac{2^n}{|Q^{(1)}|} \int_{Q^{(1)}} |f| \leq 2^n \lambda.$$
Assume that $x \in \mathbb{R}^n \setminus \Omega$. Then by the Lebesgue differentiation theorem it follows that $|g(x)| \leq \lambda$ a.e.. Thus
$$\|g\|_{L^\infty} \leq 2^n \lambda.$$
□

A short-hand CZD will stand for Calderón-Zygmund decomposition, and for a function $f \in L^1$ and $\lambda > 0$, we call the decomposition given by the previous lemma "the CZD of the function f at the level λ".

Definition 1.19. *Assuming f is a measurable function, its distribution function is defined as $\lambda_f(t) = |\{x : |f(x)| > t\}|$ and for $p > 0$ its weak L^p-norm as*
$$\|f\|_{L^{p,\infty}} = \sup_{t>0} t \lambda_f^{1/p}(t),$$
and the weak-L^p -space as
$$L^{p,\infty} = \{f \in L^0 : \|f\|_{L^{p,\infty}} < \infty\}.$$

Remark 1.20. For $1 \leq p < \infty$ we have the inclusion $L^p \subset L^{p,\infty}$.

Definition 1.21. *For some set Y, let X be the set of all functions $f : Y \longrightarrow \mathbb{C}$ and $T : X \longrightarrow X$ an operator. The operator T is said to be sublinear, if*
$$|T(f+g)(x)| \leq |Tf(x)| + |Tg(x)|$$
holds for all $x \in Y$ for all $f, g \in X$.

The following interpolation theorem known as Marcinkiewicz interpolation is often useful (see [1]).

Theorem 1.22. *Assume that the operator* $T : L^s \longrightarrow L^{s,\infty}$ *is sublinear and bounded for* $s = p, q$ *with* $1 \leq p < q < \infty$. *Then the operator* $T : L^r \longrightarrow L^r$ *is bounded for all* $p < r < q$.

Lemma 1.23. *The square operator is sub-linear.*

Proof. Follows from the triangle inequality of ℓ^2. □

Theorem 1.24. *If* $f \in L^1$, *then*

$$|\{x : S_{\mathcal{D}} f > \lambda\}| \leq A_n \frac{\|f\|_{L^1}}{\lambda},$$

where $A_n > 0$.

Proof. Fix $\lambda > 0$ and let $f = g + \sum_Q b_Q$ be the CZD of the function f at the level λ. By sub-linearity of the square operator and sub-additivity of measure

$$|\{x : S_{\mathcal{D}} f > \lambda\}| \leq |\{x : S_{\mathcal{D}} g > \frac{\lambda}{2}\}| + |\{x : S_{\mathcal{D}} b > \frac{\lambda}{2}\}| := I + II.$$

By theorem 1.18, $g \in L^1 \cap L^\infty \subset L^2$ so that by theorem 1.17 and Chebyshevs' inequality and properties of the CZD of the function f at the level λ

$$\begin{aligned} I = |\{x : S_{\mathcal{D}} g > \frac{\lambda}{2}\}| &\leq \frac{4}{\lambda^2} \int (S_{\mathcal{D}} g)^2 = \frac{4}{\lambda^2} \|S_{\mathcal{D}g}\|_{L^2}^2 \\ &\overset{1.17}{=} \frac{4}{\lambda^2} \|g\|_{L^2}^2 = \frac{4}{\lambda^2} \int |g|^2 \leq \frac{4 \cdot 2^n \lambda}{\lambda^2} \int |g| \\ &= C_n \frac{\|g\|_{L^1}}{\lambda} = C_n \frac{\|f\|_{L^1}}{\lambda}. \end{aligned}$$

To estimate II we notice that if $Q \subsetneq R$, then

$$\Delta_R b_Q = 0,$$

since $\int b_Q = 0$, and also that if $Q \cap R \neq \emptyset$, then

$$\Delta_R b_Q = 0.$$

Thus we have that $\operatorname{spt}(\Delta_R b_Q) \subset Q$ for all $R \in \mathcal{D}$ and especially that $\operatorname{spt}(\mathcal{S}_{\mathcal{D}}(b_Q)) \subset Q$. Applying this and using sublinearity of the square function we have

$$\begin{aligned} II = |\{x : S_{\mathcal{D}} b > \frac{\lambda}{2}\}| &\leq |\{x : \sum_{Q \in \Omega_\lambda} S_{\mathcal{D}} b_Q > \frac{\lambda}{2}\}| \\ &\leq |\{x : \sum_{Q \in \Omega_\lambda} S_{\mathcal{D}} b_Q > 0\}| \leq \Big| \bigcup_{Q \in \Omega_\lambda} Q \Big| \leq \frac{\|f\|_{L^1}}{\lambda}. \end{aligned}$$

Finish the proof by setting $A_n = C_n + 1$. □

Corollary 1.25. *If* $1 < p < 2$, *then* $\|S_{\mathcal{D}} f\|_{L^p} \lesssim \|f\|_{L^p}$.

Proof. By the theorems 1.24 and 1.17, the square operator is bounded as a mapping

$$S_{\mathcal{D}} : L^1 \longrightarrow L^{1,\infty}, \qquad S_{\mathcal{D}} : L^2 \longrightarrow L^{2,\infty}.$$

Then by interpolation, theorem 1.22, the square operator is bounded as a mapping

$$S_{\mathcal{D}} : L^p \longrightarrow L^p.$$

□

Accessing the estimates in the range $(2, \infty)$ for the square operator will make use of the sharp maximal function and an inequality due to Fefferman and Stein.

Definition 1.26. *Let* $f \in L^1_{loc}$. *Then the dyadic sharp maximal operator* $\mathcal{M}^{\#}_{\mathcal{D}}$ *associates to the function f the function*

$$\mathcal{M}^{\#}_{\mathcal{D}} f = \sup_{Q \in \mathcal{D}} 1_Q \langle |f - \langle f \rangle_Q| \rangle_Q.$$

For the proof of the next theorem, 1.27, also known as the Fefferman-Stein inequality see e.g. [1].

Theorem 1.27. *Let* $1 < q \leq p$, *and assume that* $\mathcal{M}_{\mathcal{D}} f \in L^q(\mathbb{R}^n)$. *Then for some* $A_{n,p}$ *we have a bound*

$$\|f\|_{L^p} \leq A_{n,p} \|\mathcal{M}^{\#}_{\mathcal{D}} f\|_{L^p}.$$

Theorem 1.28. *Let* $2 < p < \infty$ *and assume that* $f \in L^p$. *Then* $\|S_{\mathcal{D}} f\|_{L^p} \lesssim \|f\|_{L^p}$.

Proof. We let

$$\mathcal{F} = \Big\{ \sum_{i=1}^{n} \alpha_i 1_{Q_i} : \alpha_i \in \mathbb{C},\ n \in \mathbb{N},\ Q_i \in \mathcal{D} \Big\}.$$

Since the collection $\mathcal{F}$ is dense in L^p, and it is enough to prove the bound in a dense subset, we may assume that $f \in \mathcal{F}$.

It is straightforward to check that $S_{\mathcal{D}}(1_Q) \in L^p$ if $p > 1$. By sublinearity of the square function and L^p-triangle inequality it then follows that $S_{\mathcal{D}}(f) \in L^p$, if $p > 1$, for all $f \in \mathcal{F}$. By the theorem 1.12, we then have $\mathcal{M}_{\mathcal{D}}((S_{\mathcal{D}} f)^2) \in L^{p/2}$ for all $p > 2$. Thus assumptions of the theorem 1.27 are satisfied and we may apply it. Thus by theorem 1.27

$$\|S_{\mathcal{D}} f\|^p_{L^p} = \|(S_{\mathcal{D}} f)^2\|^{\frac{p}{2}}_{L^{\frac{p}{2}}} \lesssim \|\mathcal{M}^{\#}_{\mathcal{D}}((S_{\mathcal{D}} f)^2)\|^{\frac{p}{2}}_{\frac{p}{2}}.$$

Consider

$$\mathcal{M}^{\#}_{\mathcal{D}}((S_{\mathcal{D}} f)^2) = \sup_{Q \in \mathcal{D}} 1_Q \frac{1}{|Q|} \int_Q \Big| (S_{\mathcal{D}} f)^2 - \langle (S_{\mathcal{D}} f)^2 \rangle_Q \Big|.$$

Well,

$$
\begin{aligned}
&\int_Q \Big| (S_{\mathcal{D}} f)^2 - \langle (S_{\mathcal{D}} f)^2 \rangle_Q \Big| \\
=&\int_Q \Big| \sum_{R \supsetneq Q} |\Delta_R f|^2 - \langle \sum_{R \supsetneq Q} |\Delta_R f|^2 \rangle_Q + \sum_{R \subset Q} |\Delta_R f|^2 - \langle \sum_{R \subset Q} |\Delta_R f|^2 \rangle_Q \Big| \\
=&\int_Q \Big| \sum_{R \subsetneq Q} |\Delta_R f|^2 - \langle \sum_{R \subset Q} |\Delta_R f|^2 \rangle_Q \Big| \\
\leq&2 \int_Q \sum_{R \subset Q} |\Delta_R f|^2 \leq 2 \int_Q \sum_{R \in \mathcal{D}} |\Delta_R (f 1_Q)|^2 \\
\leq&\|\mathcal{S}_{\mathcal{D}}(f 1_Q)\|_{L^2}^2 = \|f 1_Q\|_{L^2}^2 = \int_Q |f|^2.
\end{aligned}
$$

By this and theorem 1.12 we have the claim:

$$
\|\mathcal{M}_{\mathcal{D}}^{\#}((S_{\mathcal{D}} f)^2)\|_{\frac{p}{2}}^{\frac{p}{2}} \lesssim \|\mathcal{M}_{\mathcal{D}} f^2\|_{\frac{p}{2}}^{\frac{p}{2}} \lesssim \|f\|_{L^p}^p.
$$

□

Collecting together theorems 1.28, 1.24 and 1.17 we have

Theorem 1.29. *If $1 < p < \infty$, then $\|S_{\mathcal{D}} f\|_{L^p} \lesssim \|f\|_{L^p}$.*

Lemma 1.30. *Let f be a measurable function and $1 < p < \infty$. Then*

$$
\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}} \leq 1} \Big| \int fg \Big|,
$$

where p, p' are dual-exponents: $1/p + 1/p' = 1$.

Proof. By Hölders' inequality

$$
\sup_{\|g\|_{L^{p'}} \leq 1} \Big| \int fg \Big| \leq \sup_{\|g\|_{L^{p'}} \leq 1} \int |fg| \leq \sup_{\|g\|_{L^{p'}} \leq 1} \|f\|_{L^p} \|g\|_{L^{p'}} \leq \|f\|_{L^p}.
$$

For the other direction we may assume that f is non-zero. Assume first that $f \in L^p$. Then let $\psi = \frac{|f|^p}{f} 1_{\{f \neq 0\}}$ and $g = \frac{\psi}{\|\psi\|_{L^{p'}}}$. Then $\|g\|_{L^{p'}} = 1$ and

$$
\int fg = \frac{\int |f|^p}{\|\psi\|_{L^{p'}}} = \frac{\int |f|^p}{(\int |f|^{(p-1)p'})^{1/p'}} = \int |f|^p \Big(\int |f|^p \Big)^{-1/p'} = \|f\|_{L^p}.
$$

Assume then that $f \notin L^p$. Let $B_n = B(0, n) \cap \{|f| \leq n\}$ and define $f_n = f 1_{B_n}$. Since $f_n \in L^p$, by the previous case there exist functions g_n such that $\|g_n\|_{L^{p'}} = 1$ and

$$
\int g_n f_n = \|f_n\|_{L^p}.
$$

Since by monotonce convergence $\|f_n\|_{L^p} \longrightarrow \|f\|_{L^p}$ by n, this gives

$$\sup_{\|g\|_{L^{p'}}\leq 1} \Big| \int fg \Big| = \infty,$$

and we are done. $\square$

Theorem 1.31. *If $1 < p < \infty$, then $\|f\|_{L^p} \lesssim \|S_{\mathcal{D}} f\|_{L^p}$.*

Proof. By lemma 1.30, the representation of the functions f, g as martingale differences, the orthogonality of the functions $\Delta_Q f$ and $\Delta_Q g$, Hölders' inequality and the reverse estimate of theorem 1.29 we get

$$\begin{aligned}
\|f\|_{L^p} &= \sup_{\|g\|_{L^{p'}}\leq 1} \Big| \int fg \Big| = \sup_{\|g\|_{L^{p'}}\leq 1} \Big| \int \sum_{R\in\mathcal{D}} \Delta_Q f \sum_{Q\in\mathcal{D}} \Delta_Q g \Big| \\
&= \sup_{\|g\|_{L^{p'}}\leq 1} \Big| \int \sum_{Q\in\mathcal{D}} \Delta_Q f \Delta_Q g \Big| \leq \sup_{\|g\|_{L^{p'}}\leq 1} \int \sum_{Q\in\mathcal{D}} |\Delta_Q f \Delta_Q g| \\
&\leq \sup_{\|g\|_{L^{p'}}\leq 1} \int S_{\mathcal{D}} f S_{\mathcal{D}} g \leq \sup_{\|g\|_{L^{p'}}\leq 1} \|S_{\mathcal{D}} f\|_{L^p} \|S_{\mathcal{D}} g\|_{L^{p'}} \\
&\lesssim \sup_{\|g\|_{L^{p'}}\leq 1} \|S_{\mathcal{D}} f\|_{L^p} \|g\|_{L^{p'}} \leq \|S_{\mathcal{D}} f\|_{L^p}.
\end{aligned}$$

$\square$

All in all we have proved

Theorem 1.32. *If $1 < p < \infty$, then*

$$\|S_{\mathcal{D}} f\|_{L^p} \sim \|f\|_{L^p}.$$

1.33 Haar basis

We define the Haar functions, the Haar basis and open their relation to martingales.

Definition 1.34. *Let $I = I^l \cup I^r \subset \mathbb{R}$ be a dyadic interval split into the two of its children and define*

$$h_I^1 = \frac{1}{\sqrt{|I|}}(1_{I^l} - 1_{I^r}), \qquad h_I^0 = \frac{1_I}{\sqrt{|I|}}.$$

Let $I = \prod_{1\leq i\leq n} I_i \subset \mathbb{R}^n$ be a dyadic interval and $\eta = (\eta_1, \ldots, \eta_n) \in \{0,1\}^n$. Then the function $h_I^\eta : I \longrightarrow \mathbb{R}$,

$$h_I^\eta(x) = \otimes_{i\leq n} h_{I_i}^{\eta_i}(x) = \prod_{1\leq i\leq n} h_{I_i}^{\eta_i}(x_i)$$

is called a Haar-function associated to the dyadic interval I, or more simply just a Haar-function, denoted with h^η. If $\eta \neq 0$, then h_I^η is called cancellative and it may be referred to as h_I. If h_I^η is not cancellative, then its non-cancellative and may be referred to as h_I^0. The collection $\{h_I^\eta : \eta \neq 0,\ I \in \mathcal{D}\}$ is called a Haar basis.

Remark 1.35. It is customary to suppress the summation, and in this thesis we also suppress the indexing, in the following ways

$$\sum_{\eta \in \{0,1\}^n \setminus \{0\}} \langle f, h_I^\eta \rangle h_I^\eta =: \sum_\eta \langle f, h_I^\eta \rangle h_I^\eta =: \langle f, h_I \rangle h_I.$$

Remark 1.36. The following holds

$$\int h_I^\eta h_K^\kappa = \begin{cases} 0 & \text{if } \eta \neq \kappa, \text{ or } \eta \neq 0 \text{ and } I \neq R, \\ 1 & \text{otherwise .} \end{cases}$$

Lemma 1.37. *Let $I \subset \mathbb{R}^n$ be a dyadic interval and let R_i, $i = 1, \ldots, 2^n$, stand for its children. Let*

$$H_I = \{f = \sum_{i=1}^{2^n} \alpha_i 1_{R_i} : \alpha_i \in \mathbb{C}, \int f = 0\}.$$

Then H_I is a vector space, and $\dim(H_I) \leq 2^n - 1$.

Proof. That H_I is vector space, is clear. Let $f = \sum_{i=1}^{2^n} \alpha_i 1_{R_i} \in H_I$. By the condition $\int f = 0$, we have that

$$\alpha_{2^n} = - \sum_{i=1}^{2^n - 1} \alpha_i.$$

Thus

$$f = \sum_{i=1}^{2^n - 1} \alpha_i (1_{R_i} - 1_{R_{2^n}})$$

and we see that $(1_{R_i} - 1_{R_{2^n}})_{i=1}^{2^n-1}$ spans H_I and thus $\dim(H_I) \leq 2^n - 1$. □

Lemma 1.38. *The sequence $\{h_I^\eta\}_{\eta \neq 0}$ is linearly independent, especially* $\dim(H_I) \geq 2^n - 1$.

Proof. It is enough to check that if

$$f = \sum_{\eta \neq 0} \alpha_\eta h_Q^\eta = 0,$$

then $\alpha_\eta = 0$ for all $\eta \neq 0$. The calculation

$$0 = \|f\|_{L^2}^2 = \int \Big| \sum_{\eta \neq 0} \alpha_\eta h_Q^\eta \Big|^2 = \int \Big(\sum_{\eta \neq 0} \alpha_\eta h_Q^\eta \Big) \Big(\sum_{\eta \neq 0} \overline{\alpha_\eta} h_Q^\eta \Big) = \sum_\eta |\alpha_\eta|^2$$

gives the claim. □

Corollary 1.39. *The space H_I is a $2^n - 1$ dimensional vector space with basis $\{h_I^\eta : \eta \neq 0\}$.*

Lemma 1.40. *Let $f, g \in L^1_{loc}$, $\eta \neq 0$ and I be a dyadic interval. Then*

1. $\Delta_I f = \Delta_I \Delta_I f$,
2. $\langle \Delta_I f, g\rangle = \langle f, \Delta_I g\rangle$,
3. $\Delta_I h_I^\eta = h_I^\eta$.

Proof. Since $\int_I \Delta_I f = 0$ and $\Delta_I f$ is constant on $R \in ch(I)$, we get the first claim:

$$\begin{aligned}\Delta_I \Delta_I f &= \sum_{R\in ch(I)} (\langle \Delta_I f\rangle_R - \langle \Delta_I f\rangle_I) 1_R = \sum_{R\in ch(I)} \langle \Delta_I f\rangle_R 1_R \\ &= \sum_{R\in ch(I)} (\langle f\rangle_R - \langle f\rangle_I) 1_R = \Delta_I f.\end{aligned}$$

For the second claim:

$$\begin{aligned}\langle \Delta_I f, g\rangle &= \sum_{R\in ch(I)} \int_R g\langle f\rangle_R - \int_I g\langle f\rangle_I = \sum_{R\in ch(I)} \langle f\rangle_R \int_R g - \langle f\rangle_I \int_I g \\ &= \sum_{R\in ch(I)} \int_R f\langle g\rangle_R - \int_I f\langle g\rangle_I = \langle f, \Delta_I g\rangle.\end{aligned}$$

Since $\int_I h_I^\eta = 0$ and h_I^η is a function that is constant on its children the last claim follows:

$$\Delta_I h_I^\eta = \sum_{R\in ch(I)} (\langle h_I^\eta\rangle_R - \langle h_I^\eta\rangle_I) 1_R = \sum_{R\in ch(I)} \langle h_I^\eta\rangle_R 1_R = h_I^\eta.$$

□

Theorem 1.41. *I being any dyadic interval, we have*

$$\Delta_I f = \sum_{\eta\neq 0} \langle f, h_I^\eta\rangle h_I^\eta.$$

Proof. Its clear that $\Delta_I f \in H_I$. Thus by corollary 1.39 there exist α_η so that

$$\Delta_I f = \sum_{\eta\neq 0} \alpha_\eta h_Q^\eta.$$

Since

$$\alpha_{\eta'} = \langle \sum_{\eta\neq 0} \alpha_\eta h_Q^\eta, h_Q^{\eta'}\rangle = \langle \Delta_I f, h_Q^{\eta'}\rangle,$$

we get

$$\Delta_I f = \sum_{\eta\neq 0} \langle \Delta_I f, h_I^\eta\rangle h_Q^\eta.$$

It remains to check that $\langle \Delta_I f, h_I^\eta\rangle = \langle f, h_I^\eta\rangle$. By Lemma 1.40 we can calculate

$$\langle f, h_I^\eta\rangle = \langle f, \Delta_I \Delta_I h_I^\eta\rangle = \langle \Delta_I f, \Delta_I h_I^\eta\rangle = \langle \Delta_I f, h_I^\eta\rangle.$$

□

Remark: Combined with the martingale representation of p-integrable functions, theorems 1.9 and 1.13, this gives a corresponding Haar basis representation of p-integrable functions, in terms of point-wise convergence a.e. for $1 \leq p < \infty$ and convergence in L^p for $1 < p < \infty$.

2 Estimates in the Banach range

Bilinear shifts and paraproducts are bilinear operators that associate to two functions f, g, a third one:

$$\Lambda(f, g).$$

The aim of this chapter is to prove estimates of the form

$$\|\Lambda(f, g)\|_{L^r} \lesssim \|f\|_{L^p} \|g\|_{L^q}$$

both for bilinear shifts and paraproducts in the Banach range, that is when $f \in L^p$, $g \in L^q$ and $1/p + 1/q = 1/r$ and $1 < p, q, r < \infty$.

Definition 2.1. *Assume that $V_i \subset L^1_{loc}$, $i = 1, 2, 3$ is some function space equipped with addition and scalar multiplication, for example L^p, L^q, L^r. An operator $\Lambda : V_1 \times V_2 \longrightarrow V_3$ is said to be bilinear, if it is linear in both variables*

$$\Lambda(f + g, h) = \Lambda(f, h) + \Lambda(g, h), \quad \Lambda(f, g + h) = \Lambda(f, g) + \Lambda(f, h),$$

and

$$\Lambda(\alpha g, \beta h) = \alpha\beta\Lambda(g, h)$$

holds for all scalars α, β.

Definition 2.2. *Let $\mathcal{F} \subset L^\infty_c$ be a function space and $\Lambda \colon \mathcal{F} \times \mathcal{F} \longrightarrow L^1_{loc}$ a bilinear map. If there exists a map $T_1 \colon \mathcal{F} \times \mathcal{F} \longrightarrow L^1_{loc}$, such that for all $f_1, f_2, f_3 \in \mathcal{F}$ it holds that*

$$\langle \Lambda(f_3 f_2), f_1 \rangle = \langle T_1(f_1, f_2), f_3 \rangle,$$

we write $\Lambda^{1*} = T$ and say that Λ^{1*} is the first adjoint of the map Λ. Similarly, if there exists a map $T_2 \colon \mathcal{F} \times \mathcal{F} \longrightarrow L^1_{loc}$, such that for all $f_1, f_2, f_3 \in \mathcal{F}$ it holds that

$$\langle \Lambda(f_1, f_3), f_2 \rangle = \langle T_2(f_1, f_2), f_3 \rangle,$$

we write $\Lambda^{2*} = T$ and say that Λ^{2*} is the second adjoint of the map Λ.

2.3 Bilinear shifts

First, we go through some lemmata.

2.3.1 Preliminary lemmata 1

We begin with a variation of theorem 1.30.

Lemma 2.4. *For* $1 < p < \infty$

$$\|(\sum_k f_k^2)^{1/2}\|_{L^p} = \sup_{\|(\sum_k g_k^2)^{1/2}\|_{L^{p'}} \le 1} \Big| \int \sum_k f_k g_k \Big|.$$

Proof. By two applications of Hölder's inequality:

$$\begin{aligned} \Big| \int \sum_k f_k g_k \Big| &\le \Big| \int (\sum_k f_k^2)^{1/2} (\sum_k g_k^2)^{1/2} \Big| \\ &\le \|(\sum_k f_k^2)^{1/2}\|_{L^p} \|(\sum_k g_k^2)^{1/2}\|_{L^{p'}} \le \|(\sum_k f_k^2)^{1/2}\|_{L^p}. \end{aligned}$$

On the other hand, by setting

$$g_k = \frac{f_k (\sum_i f_i^2)^{p/2-1}}{\|(\sum_k f_k^2)^{\frac{p-1}{2}}\|_{L^{p'}}},$$

we get

$$\int \sum_k f_k g_k = \frac{\int (\sum_k f_k^2)^{p/2}}{\|(\sum_k f_k^2)^{\frac{p-1}{2}}\|_{L^{p'}}} = \frac{\int (\sum_k f_k^2)^{p/2}}{(\int (\sum_k f_k^2)^{p/2})^{1/p'}} = \|(\sum_k f_k^2)^{1/2}\|_{L^p},$$

and

$$\|(\sum_k g_k^2)^{1/2}\|_{L^{p'}} = \frac{\|(\sum_k f_k^2)^{\frac{p-1}{2}}\|_{L^{p'}}}{\|(\sum_k f_k^2)^{\frac{p-1}{2}}\|_{L^{p'}}} = 1.$$

□

Then, some useful basic things.

Lemma 2.5. *If* $0 < p < q < \infty$, *and* $f, g \in L^1_{loc}$, *then*

$$\langle |f|^p \rangle_Q^{1/p} \le \langle |f|^q \rangle_Q^{1/q}.$$

Also the average can be moved from one function to the other as

$$\langle |f| \rangle_Q \int_Q |g| = \langle |f| \rangle_Q |Q| \langle |g| \rangle_Q = \int_Q |f| \langle |g| \rangle_Q.$$

Proof. The first claim follows by applying Hölders' inequality with the exponent q/p, the second is proved as it is stated. $\square$

The following estimate know as Stein's inequality is also used

Lemma 2.6. *Let $1 < p < \infty$. Then for measurable functions f_Q, $Q \in \mathcal{D}$*

$$\|(\sum_Q \langle f_Q \rangle_Q^2 1_Q)^{1/2}\|_{L^p} \lesssim \|(\sum_Q |f_Q|^2 1_Q)^{1/2}\|_{L^p}.$$

Proof. Three cases. Assume first that $p = 2$. Then we can estimate

$$\begin{aligned}\|(\sum_Q \langle f_Q \rangle_Q^2 1_Q)^{1/2}\|_{L^2}^2 &= \int \sum_Q \Big| \langle f_Q \rangle_Q^2 1_Q \Big| \le \sum_Q \int \langle |f_Q|^2 \rangle_Q 1_Q \\ &= \int_Q \sum_Q |f_Q|^2 = \|(\sum_Q |f_Q|^2 1_Q)^{1/2}\|_{L^2}^2\end{aligned}$$

Assume then that $p > 2$. By lemma 1.30 with integration against some function ψ with $\|\psi\|_{L^{(p/2)'}} \le 1$ we can write

$$\|(\sum_Q \langle f_Q \rangle_Q^2 1_Q)^{1/2}\|_{L^p}^2 = \|\sum_Q \langle f_Q \rangle_Q^2 1_Q\|_{L^{p/2}} = \Big| \int \sum_Q \langle f_Q \rangle_Q^2 1_Q \psi \Big|.$$

By lemma 2.5 we may continue

$$\begin{aligned}\Big| \int \sum_Q \langle f_Q \rangle_Q^2 1_Q \psi \Big| &\le \int \sum_Q \langle |f_Q|^2 \rangle_Q 1_Q |\psi| = \int \sum_Q |f_Q|^2 1_Q \langle |\psi| \rangle_Q \le \int \sum_Q |f_Q|^2 1_Q \mathcal{M}_{\mathcal{D}} \psi \\ &= \int \mathcal{M}_{\mathcal{D}} \psi \sum_Q |f_Q|^2 1_Q \le \|\mathcal{M}_{\mathcal{D}} \psi\|_{L^{(p/2)'}} \|\sum_Q |f_Q|^2 1_Q\|_{L^{p/2}} \le \|\sum_Q |f_Q|^2 1_Q\|_{L^{p/2}}.\end{aligned}$$

Thus

$$\Big\|(\sum_Q \langle f_Q \rangle_Q^2 1_Q)^{1/2}\Big\|_{L^p}^2 \le \Big\|\sum_Q |f_Q|^2 1_Q\Big\|_{L^{p/2}},$$

which gives

$$\Big\|(\sum_Q \langle f_Q \rangle_Q^2 1_Q)^{1/2}\Big\|_{L^p} \le \Big\|\Big(\sum_Q |f_Q|^2\Big)^{1/2} 1_Q\Big\|_{L^p}.$$

Last, assume that $1 < p < 2$. By lemma 2.4

$$\|(\sum_{Q \in \mathcal{D}} \langle f_Q \rangle_Q^2 1_Q)^{1/2}\|_{L^p} = \Big| \int \sum_Q \langle f_Q \rangle_Q 1_Q g_Q \Big|,$$

where $\|(\sum_{Q \in \mathcal{D}} g_Q^2)^{1/2}\|_{L^{p'}} \le 1$. We may estimate this further by lemma 2.5 and Hölders' inequality

$$\Big| \int \sum_Q \langle f_Q \rangle_Q 1_Q g_Q \Big| = \Big| \int \sum_Q f_Q 1_Q \langle g_Q \rangle \Big| \le \int \Big(\sum_Q |f_Q|^2 1_Q\Big)^{1/2} \Big(\sum_Q \langle |g_Q| \rangle_Q^2\Big)^{1/2}$$

$$\leq \Big\| \Big(\sum_Q |f_Q|^2 1_Q \Big)^{1/2} \Big\|_{L^p} \Big\| \Big(\sum_Q \langle |g_Q| \rangle_Q^2 1_Q \Big)^{1/2} \Big\|_{L^{p'}}.$$

Since $p' \in (2, \infty)$, we may apply the already proven part of this lemma with the last term:

$$\begin{aligned} \Big\| \Big(\sum_Q |f_Q|^2 1_Q \Big)^{1/2} \Big\|_{L^p} \Big\| \sum_Q \Big(\langle |g_Q| \rangle_Q^2 1_Q \Big)^{1/2} \Big\|_{L^{p'}} &\lesssim \Big\| \Big(\sum_Q |f_Q|^2 1_Q \Big)^{1/2} \Big\|_{L^p} \Big\| \sum_Q \Big(|g_Q|^2 \Big)^{1/2} \Big\|_{L^{p'}} \\ &\leq \Big\| \Big(\sum_Q |f_Q|^2 1_Q \Big)^{1/2} \Big\|_{L^p}. \end{aligned}$$

□

Now we define bilinear shifts and prove their boundedness.

2.6.1 Boundedness of bilinear shifts

Definition 2.7. *Let $f, g \in L^1_{loc}$ and $i, j, k \geq 0$. We define the cancellative bilinear shift $S^{i,j,k}$ as*

$$S^{i,j,k}(f,g) = \sum_Q A_Q^{i,j,k}(f,g),$$

where for all $Q \in \mathcal{D}$, $A_Q^{i,j,k}(f,g)$ is one of the following three

$$A_Q^{i,j,k}(f,g) = \sum_{I^i = J^j = K^k = Q} \alpha_{IJKQ} \langle f, h_I \rangle \langle g, h_J \rangle h_K^0,$$

$$A_Q^{i,j,k}(f,g) = \sum_{I^i = J^j = K^k = Q} \alpha_{IJKQ} \langle f, h_I \rangle \langle g, h_J^0 \rangle h_K,$$

$$A_Q^{i,j,k}(f,g) = \sum_{I^i = J^j = K^k = Q} \alpha_{IJKQ} \langle f, h_I^0 \rangle \langle g, h_J \rangle h_K,$$

where

$$|\alpha_{IJKQ}| \leq \frac{\sqrt{|I||J||K|}}{|Q|^2}.$$

Remark 2.8. With given parameters (i, j, k), there are three different types of shifts.

Remark 2.9. One easily checks that the adjoints of bilinear shifts are bilinear shifts.

Remark 2.10. The reader may think that $\alpha_{IJKQ} \neq 0$ for only finitely many I, J, K, Q. Then everything is well-defined from the get go, and none of our estimates depend on the number of cubes for which $\alpha_{IJKQ} \neq 0$.

What makes the shift cancellative is that two or more of the Haar functions are always cancellative and regardless of their positioning and choosing this allows us to prove

Theorem 2.11. *Let $1 < p, q, r < \infty$ and $1/p + 1/q = 1/r$. Then*

$$\|S^{i,j,k}(f,g)\|_{L^r} \lesssim \|f\|_{L^p}\|g\|_{L^q}.$$

Proof. If we can prove the claim for one of the forms a shift can take, then from duality the claim follows for its adjoints, see e.g. proof of theorem 2.29. We assume that a cancellative bilinear shift is of the form

$$S^{i,j,k}(f,g) = \sum_Q \sum_{I^i=J^j=K^k=Q} \alpha_{IJKQ}\langle f, h_I\rangle\langle g, h_J\rangle h_K^0.$$

Define

$$\Delta_Q^i f = \sum_{I^i=Q} \langle f, h_I\rangle h_I$$

and notice that by the orthogonality of Haar-functions

$$A_Q^{i,j,k}(f,g) = A_Q^{i,j,k}(\Delta_Q^i f, g) = A_Q^{i,j,k}(\Delta_Q^i f, \Delta_Q^j g).$$

By lemma 1.30

$$\|S^{i,j,k}(f,g)\|_{L^r} = \sup_{\|\psi\|_{L^{r'}}\le 1} \Big|\int S^{i,j,k}(f,g)\psi\Big|,$$

and focusing on the integral

$$\begin{aligned}
\Big|\int S^{i,j,k}(f,g)\psi\Big| &\le \sum_Q \sum_{I^i=J^j=K^k=Q} |\alpha_{IJKQ}||\langle f,h_I\rangle||\langle g,h_J\rangle||\langle \psi, h_K^0\rangle| \\
&\le \sum_Q \sum_{I^i=J^j=K^k=Q} |\alpha_{IJKQ}|\langle|\Delta_Q^i f|, |h_I|\rangle\langle|\Delta_Q^j g|, |h_J|\rangle\langle|\psi|, |h_K^0|\rangle \\
&\le \sum_Q \sum_{I^i=J^j=K^k=Q} |Q|^{-2}\int_I |\Delta_Q^i f|\int_J |\Delta_Q^j g|\int_K |\psi| \\
&= \sum_Q |Q|\langle|\Delta_Q^i f|\rangle_Q\langle|\Delta_Q^j g|\rangle_Q\langle|\psi|\rangle_Q \\
&= \int \sum_Q \langle|\Delta_Q^i f|\rangle_Q 1_Q\langle|\Delta_Q^j g|\rangle_Q\langle|\psi|\rangle_Q 1_Q \\
&\le \int \mathcal{M}_\Delta\psi\Big[\sum_Q\langle|\Delta_Q^i f|\rangle_Q^2\Big]^{1/2}\Big[\sum_Q\langle|\Delta_Q^j g|\rangle_Q^2\Big]^{1/2} \\
&\lesssim \|\mathcal{M}_{\mathcal{D}}\psi\|_{L^{r'}}\|(\sum_Q\langle|\Delta_Q^i f|\rangle_Q^2)^{1/2}\|_{L^p}\|(\sum_Q\langle|\Delta_Q^j g|\rangle_Q^2)^{1/2}\|_{L^q} \\
&\lesssim \|\psi\|_{L^{r'}}\|(\sum_Q|\Delta_Q^i f|^2)^{1/2}\|_{L^p}\|(\sum_Q|\Delta_Q^j g|^2)^{1/2}\|_{L^q} \\
&\le \|(\sum_Q|\Delta_Q f|^2)^{1/2}\|_{L^p}\|(\sum_Q|\Delta_Q g|^2)^{1/2}\|_{L^q} \\
&\lesssim \|f\|_{L^p}\|g\|_{L^q}.
\end{aligned}$$

□

2.12 Linear paraproducts

Definition 2.13. *Let* $f \in L^1_{loc}$ *and* $\alpha = (\alpha_Q)_{Q\in\mathcal{D}}$ *be a sequence satisfying* $\|\alpha\|_{BMO_{\mathcal{D}}(1)} < \infty$. *Then a paraproduct is defined as*

$$\Pi_\alpha f = \sum_{Q\in\mathcal{D}} \alpha_Q \langle f\rangle_Q h_Q.$$

or its adjoint

$$\Pi_\alpha f = \sum_{Q\in\mathcal{D}} \alpha_Q \langle f, h_Q\rangle_Q \frac{1_Q}{|Q|}.$$

Assuming the boundedness of linear paraproducts enables a proof of a boundedness of paraproducts in the bilinear setting. Therefore an investigation of linear paraproducts is good-and-well, the next aim being a proof the estimate

$$\|\Pi_\alpha f\|_{L^p} \lesssim \|\alpha\|_{BMO_{\mathcal{D}}(1)} \|f\|_{L^p}.$$

Definition 2.14. *Let* $\mathcal{F} \subset \mathcal{D}$ *and* $(\alpha_Q)_{Q\in\mathcal{F}}$ *be a sequence of complex numbers and define*

$$\|\alpha\|_{BMO_{\mathcal{F}}(p)} = \sup_{Q_0\in\mathcal{F}} \frac{1}{|Q_0|^{1/p}} \Big\| \Big(\sum_{\substack{Q\subset Q_0\\ Q\in\mathcal{F}}} \frac{|\alpha_Q|^2}{|Q|} 1_Q \Big)^{1/2} \Big\|_{L^p},$$

$$\|\alpha\|_{BMO_{\mathcal{F}}(p,\infty)} = \sup_{Q_0\in\mathcal{F}} \frac{1}{|Q_0|^{1/p}} \Big\| \Big(\sum_{\substack{Q\subset Q_0\\ Q\in\mathcal{F}}} \frac{|\alpha_Q|^2}{|Q|} 1_Q \Big)^{1/2} \Big\|_{L^{p,\infty}}.$$

Remark 2.15. The reader may think that $\alpha_Q = 0$ for only finitely many Q. Then everything is well-defined from the get go, and none of our estimates depend on the number of cubes for which $\alpha_Q \neq 0$.

2.15.1 Preliminary lemmata 2

Next theorem referred to as the John-Nirenberg inequality holds is a very useful result enabling conversion between norms depending on what might be suitable at any given situation, this is well demonstrated in the proof of theorem 3.7.

2.15.2 John-Nirenberg inequality

Theorem 2.16. *Assume that* $\mathcal{F} \subset \mathcal{D}$ *and let* $0 < p, q < \infty$. *Then*

$$\|\alpha\|_{BMO_{\mathcal{F}}(p)} \sim \|\alpha\|_{BMO_{\mathcal{F}}(q,\infty)},$$

especially

$$\|\alpha\|_{BMO_{\mathcal{F}}(p)} \sim \|\alpha\|_{BMO_{\mathcal{F}}(q)}.$$

Proof. Notice that it is enough to prove

$$\|\alpha\|_{BMO_{\mathcal{F}}(p)} \lesssim \|\alpha\|_{BMO_{\mathcal{F}}(q,\infty)}.$$

May assume that $\mathcal{F}$ is finite. Then $\|\alpha\|_{BMO_{\mathcal{F}}(p)} < \infty$. Find $Q_0 \in \mathcal{F}$, such that

$$\|\alpha\|_{BMO_{\mathcal{F}}(p)} = \frac{1}{|Q_0|^{1/p}} \Big\| \Big(\sum_{\substack{Q \subset Q_0 \\ Q \in \mathcal{F}}} \frac{|\alpha_Q|^2}{|Q|} 1_Q \Big)^{1/2} \Big\|_{L^p}.$$

Then we can write

$$|Q_0| \|\alpha\|^p_{BMO_{\mathcal{F}}(p)} = \int_{Q_0} \Big(\sum_{\substack{Q \subset Q_0 \\ Q \in \mathcal{F}}} \frac{|\alpha_Q|^2}{|Q|} 1_Q \Big)^{p/2}.$$

To estimate this, define

$$E = \Big\{ x : \Big(\sum_{Q \subset Q_0} \frac{|\alpha_Q|^2}{|Q|} 1_Q(x) \Big)^{1/2} > \lambda \|\alpha\|_{BMO_{\mathcal{F}}(q,\infty)} \Big\}$$

The set E can be estimated by its definition to

$$|E| \leq \Big(\frac{\big\| (\sum_{Q \subset Q_0} \frac{|\alpha_Q|^2}{|Q|} 1_Q)^{1/2} \big\|_{L^{q,\infty}}}{\lambda \|\alpha\|_{BMO_{\mathcal{F}}(q,\infty)}} \Big)^q \leq \Big(\frac{|Q_0|^{1/q} \|\alpha\|_{BMO_{\mathcal{F}}(q,\infty)}}{\lambda \|\alpha\|_{BMO_{\mathcal{F}}(q,\infty)}} \Big)^q = \frac{|Q_0|}{\lambda^p}.$$

Then to proceed, we express the set E as a disjoint union of maximal cubes

$$E = \bigcup I^*$$

that are chosen on the condition that

$$\Big(\sum_{\substack{Q \subset Q_0 \\ I^* \subset Q}} \frac{|\alpha_Q|^2}{|Q|} \Big)^{1/2} > \lambda \|\alpha\|_{BMO_{\mathcal{F}}(q,\infty)}.$$

Now continuing with the estimate, applying disjointness of the sets I^* and the definition of the set E, we get

$$\begin{aligned}
&\int_{Q_0} (\sum_{Q \subset Q_0} \frac{|\alpha_Q|^2}{|Q|} 1_Q)^{p/2} = \int_E (\sum_{Q \subset Q_0} \frac{|\alpha_Q|^2}{|Q|} 1_Q)^{p/2} + \int_{E^\complement} (\sum_{Q \subset Q_0} \frac{|\alpha_Q|^2}{|Q|} 1_Q)^{p/2} \\
\leq & 2^{p/2} (\sum_{I^*} \int_{I^*} (\sum_{\substack{I^* \subsetneq Q \\ Q \subset Q_0}} \frac{|\alpha_Q|^2}{|Q|} 1_Q)^{p/2} + \sum_{I^*} \int_{I^*} (\sum_{Q \subset I^*} \frac{|\alpha_Q|^2}{|Q|} 1_Q)^{p/2}) + \lambda^p |Q_0| \|\alpha\|^p_{BMO_{\mathcal{F}}(q,\infty)} \\
= & 2^{p/2} \Big(\sum_{I^*} H + \sum_{I^*} J \Big) + \lambda^p |Q_0| \|\alpha\|^p_{BMO_{\mathcal{F}}(q,\infty)}.
\end{aligned}$$

By maximality of the cubes I^* we can estimate the first integral as

$$H = \int_{I^*} (\sum_{\substack{I^* \subsetneq Q \\ Q \subset Q_0}} \frac{|\alpha_Q|^2}{|Q|} 1_Q)^{p/2} \leq \lambda^p |I^*| \|\alpha\|^p_{BMO_{\mathcal{F}}(q,\infty)},$$

and the second integral as

$$J = \int_{I^*} (\sum_{Q \subset I^*} \frac{|\alpha_Q|^2}{|Q|} 1_Q)^{p/2} = |I^*| \frac{1}{|I^*|} \int_{I^*} (\sum_{Q \subset I^*} \frac{|\alpha_Q|^2}{|Q|} 1_Q)^{p/2} \leq |I^*| \|\alpha\|^p_{BMO_{\mathcal{F}}(p)}.$$

Using these and continuing from where we left, we have

$$\begin{aligned}
&2^{p/2} \Big(\sum_{I^*} H + \sum_{I^*} J \Big) + \lambda^p |Q_0| \|\alpha\|^p_{BMO_{\mathcal{F}}(q,\infty)} \\
\leq &2^{p/2} \Big(\sum_{I^*} \lambda^p |I^*| \|\alpha\|^p_{BMO_{\mathcal{F}}(q,\infty)} + \sum_{I^*} |I^*| \|\alpha\|^p_{BMO_{\mathcal{F}}(p)} \Big) + \lambda^p |Q_0| \|\alpha\|^p_{BMO_{\mathcal{F}}(q,\infty)} \\
\leq &2^{p/2} (\lambda^p |E| \|\alpha\|^p_{BMO_{\mathcal{F}}(q,\infty)} + |E| \|\alpha\|^p_{BMO_{\mathcal{F}}(p)}) + \lambda^p |Q_0| \|\alpha\|^p_{BMO_{\mathcal{F}}(q,\infty)} \\
\leq &2^{p/2} (\lambda^p \frac{|Q_0|}{\lambda^p} \|\alpha\|^p_{BMO(q,\infty)} + \frac{|Q_0|}{\lambda^p} \|\alpha\|^p_{BMO(p)}) + \lambda^p |Q_0| \|\alpha\|^p_{BMO(q,\infty)},
\end{aligned}$$

and tidying it up:

$$(1 - \frac{2^{p/2}}{\lambda^p}) \|\alpha\|^p_{BMO_{\mathcal{F}}(p)} \leq (2^{p/2} + \lambda^p) \|\alpha\|^p_{BMO(q,\infty)}.$$

The constant acquired is independent of the collection F. Now choosing for example $\lambda = 2\sqrt{2}$ gives the claim

$$\|\alpha\|_{BMO_{\mathcal{F}(p)}} \leq C_p \|\alpha\|_{BMO_{\mathcal{F}}(q,\infty)}.$$

□

2.16.1 Stopping time construction and sparse families

A part of the proof of the estimate we are aiming at, that of theorem 2.24, will be a sorting of the cubes via a stopping time construction contained in the following paragraph. This also gives rise to one collection of sets called a sparse family.

A stopping time construction: Fix $Q_0 \in \mathcal{D}$, let $\mathcal{F}^0_f(Q_0) = \{Q_0\}$ and for $j \geq 0$ define

$$\mathcal{F}^{j+1}_f(Q_0) = \bigcup_{Q \in \mathcal{F}^j_f(Q_0)} S(Q),$$

where

$$S(Q) = \{R \subsetneq Q : \text{R is maximal s.t. } \langle |f| \rangle_R > 2 \langle |f| \rangle_Q \}.$$

Let

$$\mathcal{F}_f(Q_0) = \bigcup_j \mathcal{F}_f^j(Q_0).$$

Let $R \in \mathcal{F}_f(Q_0)$ and associate to the cube R the set

$$E_R = R \setminus \bigcup_{\substack{\tilde{R} \subsetneq R \\ \tilde{R} \in \mathcal{F}_f(Q_0)}} \tilde{R}.$$

From this definition, it clearly follows that if $R \neq R'$, then $E_R \cap E_{R'} = \emptyset$. From the definition we see that the sets $\mathcal{F}_f^i(Q_0)$ are nested union-wise, thus if $R \in \mathcal{F}_f^j(Q_0)$, then

$$\bigcup_{\substack{\tilde{R} \subsetneq R \\ \tilde{R} \in \mathcal{F}_f(Q_0)}} \tilde{R} = \bigcup_{\substack{\tilde{R} \subsetneq R \\ \tilde{R} \in \mathcal{F}_f^{j+1}(Q_0)}} \tilde{R}.$$

Since $\langle |f| \rangle_{\tilde{R}} > 2\langle |f| \rangle_R$, we have that $|\tilde{R}| \leq \frac{1}{2\langle |f| \rangle_R} \int_{\tilde{R}} |f|$. By these we can estimate

$$|E_R| = |R| - \Big| \bigcup_{\substack{\tilde{R} \subsetneq R \\ \tilde{R} \in \mathcal{F}_f(Q_0)}} \tilde{R} \Big| = |R| - \sum_{\substack{\tilde{R} \subsetneq R \\ \tilde{R} \in \mathcal{F}_f^{j+1}(Q_0)}} |\tilde{R}| \geq |R| - \sum_{\substack{\tilde{R} \subsetneq R \\ \tilde{R} \in \mathcal{F}_f^{j+1}(Q_0)}} \frac{1}{2\langle |f| \rangle_R} \int_{\tilde{R}} |f|_R \geq \frac{|R|}{2}.$$

Especially $|R| \sim |E_R|$.

Generally any "such" collection $(E_R)_{R \in \mathcal{F}}$ is called sparse:

Definition 2.17. *A collection $\mathcal{F} \subset \mathcal{D}$ is called sparse, if for all $R \in \mathcal{F}$ there exists a set $E_R \subset R$, such that $|E_R| \sim |R|$ and the collection $(E_R)_{R \in \mathcal{D}}$ is disjoint.*

We record a useful lemma.

Lemma 2.18. *Let $\psi \in L^1_{loc}$ and $p > 1$. Then for a sparse collection $\mathcal{F}$*

$$\sum_{R \in \mathcal{F}} \langle |\psi| \rangle_R^p |R| \lesssim \|\psi\|_{L^p}^p.$$

Proof. Follows from properties of the maximal function and the sparse collection $\mathcal{F}$:

$$\sum_{R \in \mathcal{F}} \langle |\psi| \rangle_R^p |R| \lesssim \sum_{R \in \mathcal{F}} \int_{E_R} (\mathcal{M}_{\mathcal{D}} f)^p \leq \|\psi\|_{L^p}^p.$$

□

Sparse collections have a nice property - a variant of Pythagoras' theorem. The proof is contained in [2]. Next we introduce some new notation.

If $\mathcal{F}$ is some sub-collection of a dyadic grid $\mathcal{D}$, then on the condition that it exists, for all $Q \in \mathcal{F}$ we define $\Pi_{\mathcal{F}} Q$ as the unique smallest cube R in $\mathcal{F}$ that contains the cube Q. To present the proof, we also introduce another operator, a sort of cut-off martingale difference.

Definition 2.19. *Let $\mathcal{F}$ be a sparse collection and for each $S \in \mathcal{F}$ define an operator $P_S f$*

$$P_S f = \sum_{\substack{Q \in \mathcal{D} \\ \Pi_{\mathcal{F}} Q = S}} \Delta_Q f.$$

In the next lemma, we let

$$E_S = S \setminus \bigcup_{R \in ch_{\mathcal{F}} S} R.$$

Lemma 2.20. *The operator P_S satisfies*

1. $P_S f = \sum_{R \in ch_{\mathcal{F}}(S)} \langle f \rangle_R 1_R + f 1_{E_S} - \langle f \rangle_S 1_S$ *a.e.*
2. $P_S f = f$ *if and only if* f *is supported on* S, *is constant on* $ch_{\mathcal{F}} S$ *and* $\int_S f = 0$.
3. $P_S^2 f = P_S f$, *and if* $S \neq T$, *then* $P_S P_T f = 0$.
4. $\int g P_S f = \int P_S g f$ *for measurable functions* f *and* g.
5. $\|P_S f\|_{L^p} \leq 2 \|f 1_S\|_{L^p}$, *for* $1 \leq p < \infty$.

Proof. 1. $P_S f$ is supported on S which partitions to $ch_{\mathcal{F}}(S)$ and E_S. If $x \in E_S$, then the sum telescopes to $P_S f(x) = f(x) - \langle f \rangle_S$ a.e.. If $x \in R \in ch_{\mathcal{F}}(S)$, then the sum telescopes to $P_S f(x) = \langle f \rangle_R - \langle f \rangle_S$.

2. If $P_S f = f$, then by 1 we see that f is constant on $ch_{\mathcal{F}} S$ and that it is clearly supported on S. Since $\int \Delta_Q f = 0$, also $\int P_S f = 0$, and thus

$$\int_S f = \int_S P_S f = 0.$$

On the other hand, if f is supported on S, is constant on $ch_{\mathcal{F}} S$ and $\int_S f = 0$, then

$$f = f 1_S = \sum_{R \in ch_{\mathcal{F}}(S)} \langle f \rangle_R 1_R + f 1_{E_S} = \sum_{R \in ch_{\mathcal{F}}(S)} \langle f \rangle_R 1_R + f 1_{E_S} + \langle f \rangle_S 1_S.$$

3. That $P_S^2 f = Pf$ holds, follows from 2 since $P_S f$ is supported on S, is constant on $ch_{\mathcal{F}} S$ and $\int_S f = 0$.

If $S \cap T = \emptyset$, then the claim holds trivially.

Assume then that $S \subsetneq T$. Then, since $P_T f$ is constant on $ch_{\mathcal{F}} T$, it is constant on S, and we see from the definition that $P_S P_T f = 0$.

Assume then that $T \subsetneq S$. Then as $T \cap E_S = \emptyset$, $P_T f 1_{E_S} = 0$ and as $\int_Q P_T f = 0$ for all $Q \supset T$, by 1 we see that

$$P_S P_T f = \sum_{R \in ch_{\mathcal{F}}(S)} \langle P_T f \rangle_R 1_R - \langle P_T f \rangle_S 1_S = 0.$$

4. Follows from 1(or definition) with the standard trick of moving the average from the function f to the function g.

5. Follows by 1:

$$\|P_S f\|_{L^p} \leq \Big\| \sum_{R \in ch_{\mathcal{F}}(S)} \langle f \rangle_R 1_R + f 1_{E_S} \Big\|_{L^p} + \|f 1_S\|_{L^p},$$

and lemma 2.5:

$$\begin{aligned} \Big\| \sum_{R \in ch_{\mathcal{F}}(S)} \langle f \rangle_R 1_R + f 1_{E_S} \Big\|_{L^p}^p &= \int \sum_{R \in ch_{\mathcal{F}}(S)} |\langle f \rangle_R 1_R|^p + |f 1_{E_S}|^p \\ &\leq \int \sum_{R \in ch_{\mathcal{F}}(S)} \langle |f|^p \rangle_R 1_R + |f 1_{E_S}|^p \\ &= \int |f|^p 1_S = \|f 1_S\|_{L^p}^p. \end{aligned}$$

□

Yet another version of lemma 1.30 will provide useful

Lemma 2.21. *For integrable functions f_k*

$$\Big(\sum_k \|f_k\|_{L^p}^p\Big)^{1/p} = \sup_{\sum_k \|g_k\|_{L^{p'}}^{p'} \leq 1} \Big| \sum_k \int f_k g_k \Big|.$$

Proof. By two applications Hölder's inequality one can see the direction " $\geq$ ". By lemma 1.30 for some unitary functions $\psi_k \in L^{p'}$, $\|f_k\|_{L^p} = \int f_k \psi_k$. Summing over k and modifying suitably gives

$$\Big(\sum_k \|f\|_p^p\Big)^{1/p} = \sum_k \int f_k \frac{\psi_k \|f_k\|_{L^{p-1}}}{((\sum_k \|f_k\|_{L^p}^p))^{1/p'}}.$$

Now choosing

$$g_k = \frac{\psi_k \|f_k\|_{L^{p-1}}}{((\sum_k \|f_k\|_{L^p}^p))^{1/p'}}$$

and checking the condition $\sum_k \|g_k\|_{L^{p'}}^{p'} \leq 1$ gives the direction " $\leq$ ". □

Theorem 2.22. *Let $1 < p < \infty$ and $\mathcal{F}$ be a sparse collection and for each $Q \in \mathcal{F}$ let f_Q be a function supported on Q that is constant on its children $ch_{\mathcal{F}}(Q)$. Then*

$$\| \sum_{Q \in \mathcal{F}} f_Q \|_{L^p} \lesssim \Big(\sum_{Q \in \mathcal{F}} \|f_Q\|_{L^p}^p \Big)^{1/p}.$$

Moreover if $\int_Q f_Q = 0$, then the reverse estimate holds.

Proof. By duality, lemma 1.30, it is enough to estimate

$$\Big|\int \sum_{Q\in\mathcal{F}} f_Q\psi\Big|$$

for ψ such that $\|\psi\|_{L^{p'}} \leq 1$.

As f_Q is supported on Q, as Q is partitioned by $ch_{\mathcal{F}}(Q)$ and E_Q, and as f_Q is constant on its children we acquire

$$\Big|\int \sum_Q f_Q\psi\Big| = \Big|\sum_Q \int_Q f_Q\psi\Big| \leq \Big|\sum_Q \sum_{R\in ch_{\mathcal{F}}(Q)} \langle f_Q\rangle_R \int_R \psi\Big| + \Big|\int \sum_Q 1_{E_Q} f_Q g\Big|.$$

We estimate the first summand by lemma 2.18

$$\begin{aligned}
&\Big|\sum_Q \sum_{R\in ch_{\mathcal{F}}(Q)} \langle f_Q\rangle_R \int_R \psi\Big| \leq \sum_Q \sum_{R\in ch_{\mathcal{F}}(Q)} |\langle f_Q\rangle|_R |R|^{1/p} \langle|\psi|\rangle_R |R|^{1/p'}\\
\leq&\Big(\sum_Q \sum_{R\in ch_{\mathcal{F}}(Q)} \langle|f_Q|\rangle_R^p |R|\Big)^{1/p}\Big(\sum_Q \sum_{R\in ch_{\mathcal{F}}(Q)} \langle|\psi|\rangle_R^{p'} |R|\Big)^{1/p'}\\
\leq&\Big(\sum_Q \sum_{R\in ch_{\mathcal{F}}(Q)} \langle|f_Q|^p\rangle_R |R|\Big)^{1/p}\|\psi\|_{L^{p'}}\\
=&\Big(\sum_Q \sum_{R\in ch_{\mathcal{F}}(Q)} \int_R |f_Q|^p\Big)^{1/p}\|\psi\|_{L^{p'}}\\
\leq&\Big(\sum_Q \int_Q |f_Q|^p\Big)^{1/p}\|\psi\|_{L^{p'}} \lesssim (\sum_Q \|f_Q\|_{L^p}^p)^{1/p}\|\psi\|_{L^{p'}}
\end{aligned}$$

By the disjointness of E_R we can estimate the second summand:

$$\begin{aligned}
\int \sum_Q 1_{E_Q} f_Q\psi &\leq \|\sum_Q 1_{E_Q} f_Q\|_{L^p}\|\psi\|_{L^{p'}}\\
&= (\sum_Q \|1_{E_Q} f_Q\|_{L^p}^p)^{1/p}\|\psi\|_{L^{p'}} \leq (\sum_Q \|f_Q\|_{L^p}^p)^{1/p}\|\psi\|_{L^{p'}}.
\end{aligned}$$

Combining the estimates and using the assumption that $\|\psi\|_{L^{p'}} \leq 1$ gives

$$\Big|\int \sum_Q f_Q\psi\Big| \lesssim (\sum_Q \|f_Q\|_{L^p}^p)^{1/p}\|\psi\|_{L^{p'}} \lesssim (\sum_Q \|f_Q\|_{L^p}^p)^{1/p}.$$

Next we prove that

$$(\sum_{Q\in\mathcal{F}} \|f_Q\|_{L^p}^p)^{1/p} \lesssim \|\sum_{Q\in\mathcal{F}} f_Q\|_{L^p}$$

under the assumption that $\int_Q f_Q = 0$. By lemma 2.21, it is enough to prove

$$\Big| \sum_{Q\in\mathcal{F}} \int f_Q g_Q \Big| \lesssim \| \sum_{Q\in\mathcal{F}} f_Q \|_{L^p} \sum_{Q\in\mathcal{F}} \|g_Q\|_{L^{p'}}^{p'}$$

for $g_Q \in L^{p'}$ that satisfy

$$\sum_{Q\in\mathcal{F}} \|g_Q\|_{L^{p'}}^{p'} \leq 1.$$

We apply the properties of the operator P_S recorded in lemma 2.20 to the functions f_Q and g_Q. We get

$$\begin{aligned}\sum_{Q\in\mathcal{F}} \int f_Q g_Q \overset{2.}{=} \sum_{Q\in\mathcal{F}} \int P_Q^2 f_Q g_Q \overset{4.}{=} \sum_{Q\in\mathcal{F}} \int P_Q f_Q P_Q g_Q \\ \overset{2.}{=} \sum_{Q\in\mathcal{F}} \int P_Q f_Q \sum_{R\in\mathcal{F}} P_R g_R \overset{2.}{=} \sum_{Q\in\mathcal{F}} \int f_Q \sum_{R\in\mathcal{F}} P_R g_R.\end{aligned}$$

The functions $P_R g_R$ are supported on R and constant on $ch_{\mathcal{F}}(R)$. Thus the assumptions for the already proven estimate are satisfied by $P_R g_R$ and by this and Hölder's inequality and property 5. of lemma 2.20 we have the claim:

$$\begin{aligned}&\Big| \sum_{Q\in\mathcal{F}} \int f_Q \sum_{R\in\mathcal{F}} P_R g_R \Big| \leq \int \Big| \sum_{Q\in\mathcal{F}} f_Q \sum_{R\in\mathcal{F}} P_R g_R \Big| \leq \| \sum_{Q\in\mathcal{F}} f_Q \|_{L^p} \| \sum_{R\in\mathcal{F}} P_R g_R \|_{L^{p'}} \\ &\lesssim \| \sum_{Q\in\mathcal{F}} f_Q \|_{L^p} (\sum_{R\in\mathcal{F}} \|P_R g_R\|_{L^{p'}}^{p'})^{1/p'} \lesssim \| \sum_{Q\in\mathcal{F}} f_Q \|_{L^p} (\sum_{R\in\mathcal{F}} \|g_R\|_{L^{p'}}^{p'})^{1/p'}.\end{aligned}$$

□

2.22.1 Boundedness of linear paraproducts

Before giving the proof, we notice that it is enough to prove the estimate restricted to those cubes contained in a fixed top cube Q_0. This goes hand in hand with the previous stopping time construction.

Lemma 2.23. *Let $\mathcal{F} \subset \mathcal{D}$ be a collection of dyadic cubes and assume that for all $Q_0 \in \mathcal{F}$ we can prove an estimate of the form*

$$\sum_{\substack{Q\in\mathcal{F} \\ Q\subset Q_0}} \alpha_Q \leq \beta \|f 1_{Q_0}\|_{L^p} \|g 1_{Q_0}\|_{L^q}.$$

Then actually

$$\sum_{Q\in\mathcal{F}} \alpha_Q \leq \beta \|f\|_{L^p} \|g\|_{L^q}.$$

Proof. Lets define $\tilde{B}_0(R) = \{Q \in \mathcal{F} : \text{ maximal } Q \text{ s.t. } Q \subset B(0,R)\}$. Then first by using the assumption, then by Hölder's inequality and last by the disjointness of maximal cubes we have the claim:

$$\begin{aligned}
\sum_{Q\in\mathcal{F}} \alpha_Q &= \lim_{R\to\infty} \sum_{Q_0\in\tilde{B}_0(R)} \sum_{Q\subset Q_0} \alpha_Q \le \beta \lim_{R\to\infty} \sum_{Q_0\in\tilde{B}_0(R)} \|f1_{Q_0}\|_{L^p}\|g1_{Q_0}\|_{L^q} \\
&\le \beta \lim_{R\to\infty} \Big(\sum_{Q_0\in\tilde{B}_0(R)} \|f1_{Q_0}\|_{L^p}^p\Big)^{1/p}\Big(\sum_{Q_0\in\tilde{B}_0(R)} \|g1_{Q_0}\|_{L^q}^q\Big)^{1/q} \\
&\le \beta \lim_{R\to\infty} (\|f1_{\tilde{B}_0(R)}\|_{L^p}^p)^{1/p}(\|g1_{\tilde{B}_0(R)}\|_{L^q}^q)^{1/q} \\
&= \beta\|f\|_{L^p}\|g\|_{L^q}.
\end{aligned}$$

□

We are finally ready to prove the boundedness of linear paraproducts.

Theorem 2.24. *Let $1 < p < \infty$. Everything being as before, we have that*

$$\|\Pi_\alpha f\|_{L^p} \lesssim \|\alpha\|_{BMO_{\mathcal{D}}(1)}\|f\|_{L^p}.$$

Proof. By lemma 1.30, it is enough to consider

$$\Big|\langle \Pi_\alpha f, \psi\rangle\Big| = \Big|\sum_{Q\in\mathcal{D}} \alpha_Q \langle f\rangle_Q \langle \psi, h_Q\rangle\Big| \le \sum_{Q\in\mathcal{D}} |\alpha_Q||\langle f\rangle_Q|\langle \psi, h_Q\rangle|,$$

where $\psi \in L^{p'}$, and more, by lemma 2.23 its enough to show the bound

$$\sum_{\substack{Q\subset Q_0\\ Q\in\mathcal{D}}} |\alpha_Q|\langle f\rangle_Q|\langle \psi, h_Q\rangle| \le \|\alpha\|_{BMO_{\mathcal{D}}(1)}\|f1_{Q_0}\|_{L^p}\|g1_{Q_0}\|_{L^q}$$

for a fixed $Q_0 \in \mathcal{D}$. By the stopping time construction we acquire

$$\sum_{\substack{Q\subset Q_0\\ Q\in\mathcal{D}}} |\alpha_Q|\langle f\rangle_Q|\langle \psi, h_Q\rangle| = \sum_{R\in\mathcal{F}_f(Q_0)} \sum_{\substack{Q\in\mathcal{D}\\ \Pi_{\mathcal{F}_f(Q_0)}Q=R}} |\alpha_Q|\langle |f|\rangle_Q|\langle \psi, h_Q\rangle|,$$

where $\Pi_{\mathcal{F}_f(Q_0)}Q$ is the smallest cube in the collection $\mathcal{F}_f(Q_0)$ strictly containing Q. From now on in this proof we write $\Pi_{\mathcal{F}_f(Q_0)}Q$ as ΠQ. Continuing with the stopping time construction gives

$$\sum_{R\in\mathcal{F}_f(Q_0)} \sum_{\substack{Q\in\mathcal{D}\\ \Pi Q=R}} |\alpha_Q|\langle |f|\rangle_Q|\langle \psi, h_Q\rangle| \lesssim \sum_{R\in\mathcal{F}_f(Q_0)} \langle |f|\rangle_R \sum_{\substack{Q\in\mathcal{D}\\ \Pi Q=R}} |\alpha_Q||\langle \psi, h_Q\rangle|.$$

Now focus on the innermost sum. Let $\psi_R = \sum_{\substack{Q\in\mathcal{D}\\ \Pi Q=R}} \Delta_Q\psi$. Then by orthogonality of Haar-functions, lemma 1.41, we have the equality, after which two applications of Hölders inequality and lastly the square function estimate of theorem 1.32 gives

$$\sum_{\substack{Q\in\mathcal{D}\\ \Pi Q=R}} |\alpha_Q||\langle\psi,h_Q\rangle| = \sum_{\substack{Q\in\mathcal{D}\\ \Pi Q=R}} |\alpha_Q||\langle\psi_R,h_Q\rangle| \lesssim \int \sum_{\substack{Q\in\mathcal{D}\\ \Pi Q=R}} |\alpha_Q||\langle\psi_R,h_Q\rangle|\frac{1_Q}{|Q|}$$

$$\leq \int \Big(\sum_{\substack{Q\in\mathcal{D}\\ \Pi Q=R}} \frac{|\alpha_Q|^2}{|Q|}1_Q\Big)^{\frac{1}{2}}\Big(\sum_{\substack{Q\in\mathcal{D}\\ \Pi Q=R}} |\langle\psi_R,h_Q\rangle|^2\frac{1_Q}{|Q|}\Big)^{\frac{1}{2}}$$

$$\leq \Big\|\Big(\sum_{\substack{Q\in\mathcal{D}\\ Q\subset R}} \frac{|\alpha_Q|^2}{|Q|}1_Q\Big)^{1/2}\Big\|_{L^p}\Big\|\Big(\sum_{Q\in\mathcal{D}} |\langle\psi_R,h_Q\rangle|^2\frac{1_Q}{|Q|}\Big)^{\frac{1}{2}}\Big\|_{L^{p'}}$$

$$\lesssim |R|^{1/p}\|\alpha\|_{BMO_{\mathcal{D}}(p)}\|\psi_R\|_{L^{p'}}.$$

Plugging this in gives

$$\sum_{\substack{Q\in\mathcal{D}\\ Q\subset Q_0}} |\alpha_Q|\langle f\rangle_Q|\langle\psi,h_Q\rangle| \lesssim \|\alpha\|_{BMO_{\mathcal{D}(p)}} \sum_{R\in\mathcal{F}_f(Q_0)} \langle|f|\rangle_R|R|^{1/p}\|\psi_R\|_{L^{p'}}$$

$$\leq \|\alpha\|_{BMO_{\mathcal{D}(p)}}\Big(\sum_{R\in\mathcal{F}_f(Q_0)} \langle|f|\rangle_R^p|R|\Big)^{1/p}\Big(\sum_{R\in\mathcal{F}_f(Q_0)} \|\psi_R\|_{L^{p'}}^{p'}\Big)^{1/p'}.$$

By lemma 2.18 we can estimate the middle term as

$$\Big(\sum_{R\in\mathcal{F}_f(Q_0)} \langle|f|\rangle_R^p|R|\Big)^{1/p} = \Big(\sum_{R\in\mathcal{F}_f(Q_0)} \langle|f1_{Q_0}|\rangle_R^p|R|\Big)^{1/p} \leq \|f1_{Q_0}\|_{L^p}$$

To estimate the last term notice that $\psi_R = \sum_{\substack{Q\in\mathcal{D}\\ \Pi Q=R}} \Delta_Q\psi$ is a function supported on R that is constant on $ch_{\mathcal{F}}(R)$, for which $\int_R \psi_R = 0$ as in theorem 2.22. Thus by theorem 2.22

$$\Big(\sum_{R\in\mathcal{F}_f(Q_0)} \|\psi_R\|_{L^{p'}}^{p'}\Big)^{1/p'} \sim \Big\|\sum_{R\in\mathcal{F}_f(Q_0)} \psi_R\Big\|_{L^{p'}}.$$

The right-hand side can be estimated with

$$\sum_{R\in\mathcal{F}_f(Q_0)} \psi_R = \sum_{R\in\mathcal{F}_f(Q_0)} \sum_{\substack{Q\in\mathcal{D}\\ \Pi Q=R}} \Delta_Q\psi = \sum_{\substack{Q\in\mathcal{D}\\ Q\subset Q_0}} \Delta_Q\psi = (\psi - \langle\psi\rangle_{Q_0})1_{Q_0}$$

and with this and lemma 2.5

$$\Big\|\sum_{R\in\mathcal{F}_f(Q_0)} \psi_R\Big\|_{L^{p'}} = \|(\psi-\langle\psi\rangle_{Q_0})1_{Q_0}\|_{L^{p'}} \leq \|\psi 1_{Q_0}\|_{L^{p'}} + \|\langle\psi\rangle_{Q_0}1_{Q_0}\|_{L^{p'}}$$

$$\leq \|\psi 1_{Q_0}\|_{L^{p'}} + \Big(\langle|\psi|^{p'}\rangle_{Q_0}|Q_0|\Big)^{1/p'}$$

$$=\|\psi 1_{Q_0}\|_{L^{p'}} + \|\psi 1_{Q_0}\|_{L^{p'}} \lesssim \|\psi 1_{Q_0}\|_{L^{p'}}.$$

Putting the pieces together gives the claim

$$\sum_{\substack{Q\subset Q_0\\ Q\in\mathcal{D}}} |\alpha_Q| \langle f\rangle_Q |\langle \psi, h_Q\rangle| \lesssim \|\alpha\|_{BMO_{\mathcal{D}(p)}} \|f 1_{Q_0}\|_{L^p} \|\psi 1_{Q_0}\|_{L^{p'}}.$$

$\square$

2.25 Bilinear paraproducts

Our next goal will be to obtain a similiar bound for bilinear paraproducts. Paraproduct is akin to a shift of complexity $(0,0,0)$, with the exception that there is less cancellativity, only one cancellative Haar function. For paraproducts to be useful the proof of theorem 2.11 then points in the direction of having to make a stricter requirement on what comes to the coefficients α_Q.

Definition 2.26. *Let $f, g \in L^1_{loc}$ and $\alpha = (\alpha_Q)_{Q\in\mathcal{D}}$ be a sequence such that $\|\alpha\|_{BMO_{\mathcal{D}}(1)} \leq \infty$. Then a bilinear paraproduct $\Pi_\alpha(f,g)$ is of one of the following three forms*

1. $\Pi_\alpha(f,g) = \Pi^1_\alpha(f,g) = \sum_{Q\in\mathcal{D}} \alpha_Q \langle f\rangle_Q \langle g\rangle_Q h_Q$,
2. $\Pi_\alpha(f,g) = \Pi^2_\alpha(f,g) = \sum_{Q\in\mathcal{D}} \alpha_Q \langle f\rangle_Q \langle g, h_Q\rangle \frac{1_Q}{|Q|}$,
3. $\Pi_\alpha(f,g) = \Pi^3_\alpha(f,g) = \sum_{Q\in\mathcal{D}} \alpha_Q \langle f, h_Q\rangle \langle g\rangle_Q \frac{1_Q}{|Q|}$.

Remark 2.27. Adjoints of bilinear paraproducts are bilinear paraproducts.

Remark 2.28. The reader may think that $\alpha_Q \neq 0$ for only finitely many Q. Then everything is well-defined from the get go, and none of our estimates depend on the number of cubes for which $\alpha_Q \neq 0$.

Theorem 2.29. *Assume that $p, q, r > 1$ and let $1/p + 1/q = 1/r$. Then*

$$\|\Pi_\alpha(f,g)\|_{L^r} \lesssim_\alpha \|f\|_{L^p}\|g\|_{L^q}.$$

Proof. To leave no room for confusion, in this proof we denote the linear paraproduct by π instead of Π.

Lets first prove this for a paraproduct of form 1. By theorems 1.32 and 1.41

$$\begin{aligned}
\|\Pi_\alpha(f,g)\|_{L^r} \sim \|S_{\mathcal{D}}\Pi_\alpha(f,g)\|_{L^r} &\leq \|(\sum_Q |\alpha_Q|^2 \langle |f|\rangle_Q^2 \langle |g|\rangle_Q^2 \frac{1_Q}{|Q|})^{1/2}\|_{L^r} \\
&= \|(\sum_Q |\alpha_Q|^2 \langle \langle |f|\rangle_Q \langle |g|\rangle_Q\rangle_Q^2 \frac{1_Q}{|Q|})^{1/2}\|_{L^r} \\
&\leq \|(\sum_Q |\alpha_Q|^2 \langle \mathcal{M}_{\mathcal{D}}f\mathcal{M}_{\mathcal{D}}g\rangle_Q^2 \frac{1_Q}{|Q|})^{1/2}\|_{L^r} \\
&= \|S_{\mathcal{D}}\pi_\beta(\mathcal{M}_{\mathcal{D}}f\mathcal{M}_{\mathcal{D}}g)\|_{L^r} \sim \|\pi_\beta(\mathcal{M}_{\mathcal{D}}f\mathcal{M}_{\mathcal{D}}g)\|_{L^r} \\
&\lesssim_\alpha \|\mathcal{M}_{\mathcal{D}}f\mathcal{M}_{\mathcal{D}}g\|_{L^r} \lesssim \|f\|_{L^p}\|g\|_{L^q},
\end{aligned}$$

where $\beta := (|\alpha_I|)_{I\in\mathcal{D}}$.

Lets then prove the estimate for a paraproduct of form 2. We could prove this as above, with only very slight modifications, but lets instead give a typical deduction of the fact that boundedness of an operator implies in some situations, like ours, the boundedness of its adjoints. Consider

$$\Pi_\alpha^2(f,g) = \sum_{Q\in\mathcal{D}} \alpha_Q \langle f\rangle_Q \langle g, h_I\rangle \frac{1_Q}{|Q|}.$$

By lemma 1.30, it is enough to estimate

$$\left|\langle \Pi_\alpha^2(f,g), \psi\rangle\right|,$$

for functions ψ such that $\|\psi\|_{L^{r'}} \le 1$. Notice first that

$$\begin{aligned}
\langle \Pi_\alpha^2(f,g), \psi\rangle &= \sum_{Q\in\mathcal{D}} \alpha_Q \langle f\rangle_Q \langle g, h_I\rangle \frac{\langle \psi, 1_Q\rangle}{|Q|} \\
&= \sum_{Q\in\mathcal{D}} \alpha_Q \langle f\rangle_Q \langle \psi\rangle_Q \langle g, h_I\rangle = \langle \Pi_\alpha^1(f,\psi), g\rangle.
\end{aligned}$$

By the relation $1/p + 1/q = 1/r$ we have that

$$\frac{1}{q'} = 1 - \frac{1}{q} = 1 - \frac{1}{r} + \frac{1}{p} = \frac{1}{r'} + \frac{1}{p},$$

so that by applying the already proven estimate:

$$\left|\langle \Pi_\alpha^1(f,\psi), g\rangle\right| \le \|\Pi_\alpha^1(f,\psi)\|_{L^{q'}} \|g\|_{L^q} \le \|f\|_{L^p} \|\psi\|_{L^{r'}} \|g\|_{L^q} \le \|f\|_{L^p} \|g\|_{L^q}.$$

Similarly for a paraproduct of form 3. $\square$

3 Estimates in the quasi-Banach range

First, a form of dualization. Notice in the following proof that if $r < 1$, then $r' < 0$.

3.1 A dualization lemma

Lemma 3.2. *Let $0 < r < \infty$. The following are equivalent:*

1. $\|f\|_{L^{r,\infty}} \lesssim A$.
2. *If $0 < |E| < \infty$, then there exists $E' \subset E$ such that $|E'| \geq |E|/2$, and*
$$|\langle f, \psi \rangle| \lesssim A|E|^{1/r'},$$
for all ψ such that $|\psi| \leq 1_{E'}$, where $1/r + 1/r' = 1$.

Proof. Assume 1. and define
$$D = \{|f| > \eta A|E|^{-1/r}\}.$$
By assumption we have
$$|D| \leq (\frac{CA}{\eta A|E|^{-1/r}})^r = \frac{C^r|E|}{\eta^r},$$
for some $C > 0$. Then by letting $E'_\eta = E \setminus D$ we have
$$(1 - (\frac{C}{\eta})^r)|E| \leq |E'_\eta|.$$
Checking the other condition with E'_η and $|\psi| \leq 1_{E'_\eta}$ gives
$$|\langle f, \psi \rangle| \leq \int_{E'_\eta} |f| \leq |E'_\eta|\eta A|E|^{-1/r} \leq \eta A|E|^{1/r'}.$$
Now choosing, for example,
$$\eta = C2^{1/r}$$
gives 2.

Assume 2. and let $E = \{|f| > \lambda\}$. Then as $\left|\frac{\overline{f}}{|f|}1_{E'}1_{\{f\neq 0\}}\right| \leq 1_{E'}$,
$$|E| \lesssim \frac{1}{\lambda}\int_{E'} |f| = \frac{1}{\lambda}\int f\frac{\overline{f}}{|f|}1_{E'}1_{\{f\neq 0\}} \lesssim \frac{1}{\lambda}A|E|^{1/r'}.$$
Rearranging gives
$$|E| \lesssim (\frac{A}{\lambda})^{1/r}$$
and thus
$$\|f\|_{L^{r,\infty}} \lesssim A.$$
□

Remark 3.3. To compare lemma 3.2 with lemma 1.30: They both give access to forms from which one can start to estimate upwards, namely $|\langle f, \psi \rangle|$.

Next we prove

3.4 A weak end-point estimate for paraproducts

Theorem 3.5. *Assume that $f, g \in L^1$. Then*

$$\|\Pi_\alpha(f,g)\|_{L^{1/2,\infty}} \lesssim \|\alpha\|_{BMO_{\mathcal{D}}(1)} \|f\|_{L^1} \|g\|_{L^1}.$$

Proof. We have three cases to prove, since $\Pi_\alpha(f,g)$ can be of three forms. Of the forms a paraproduct can take, 2. and 3. are similar. It's then enough to prove 1. and 2.. Assume first that

$$\Pi_\alpha(f,g) = \sum_{Q \in \mathcal{D}} \alpha_Q \langle f \rangle_Q \langle g \rangle_Q h_Q.$$

By scaling we may assume that $\|f\|_{L^1} = \|g\|_{L^1} = 1$. By lemma 3.2, for an arbitrary set E for which $0 < |E| < \infty$ it is enough to find a set $E' \subset E$ such that $|E| \lesssim |E'|$ and show that

$$|\langle \Pi_\alpha(f,g), \psi \rangle| \lesssim |E|^{-1},$$

where ψ is such that $|\psi| \leq 1_{E'}$.

Fix the set E and let

$$\Omega = \{\mathcal{M}_\Delta f > C|E|^{-1}\} \cup \{\mathcal{M}_\Delta g > C|E|^{-1}\}.$$

As $f, g \in L^1$ choose $C > 0$ independent of the set $|E|$, such that $|\Omega| < \frac{|E|}{2}$ and let

$$E' = E \setminus \Omega.$$

Then $|E'| \geq \frac{|E|}{2}$.

It remains to check the condition $|\langle \Pi_\alpha(f,g), \psi \rangle| \lesssim |E|^{-1}$. As ψ is supported on E', we have

$$|\langle \Pi_\alpha(f,g), \psi \rangle| \leq \sum_{Q \in \mathcal{D}} |\alpha_Q| \langle |f| \rangle_Q \langle |g| \rangle_Q |\langle \psi, h_Q \rangle| = \sum_{\substack{Q \in \mathcal{D} \\ Q \cap \Omega^{\complement} \neq \emptyset}} |\alpha_Q| \langle |f| \rangle_Q \langle |g| \rangle_Q |\langle \psi, h_Q \rangle|.$$

To split the sum into manageable parts, we consider the collections of maximal cubes

$$\mathcal{F}_m(\phi) = \{Q_{MAX} : Q_{MAX} \text{ is maximal s.t. } \langle |\phi| \rangle_{Q_{MAX}} > 2^{-m} C|E|^{-1}\}, \qquad \phi = f, g,$$

and

$$\tilde{P}_m = \{I \in \mathcal{D} : I \subset Q, \text{ for some } Q \in \mathcal{F}_m(f) \cup \mathcal{F}_m(g)\},$$

and let

$$P_m = \tilde{P}_m \setminus \tilde{P}_{m-1}.$$

If $Q \cap \Omega^{\complement} \neq \emptyset$, then for all cubes $R \supset Q$ it holds that

$$\langle |f| \rangle_R \leq C|E|^{-1}$$

i.e. $R \notin \mathcal{F}_0(f)$. Similarly we have that $R \notin \mathcal{F}_0(g)$. Thus $Q \notin \tilde{P}_0$.

Assume then that $\langle |f| \rangle_Q > 0$, or $\langle |g| \rangle_Q > 0$, and $Q \cap \Omega^{\complement} \neq \emptyset$. Then exists the smallest positive integer $m \geq 1$ such that $Q \in \tilde{P}_n$ for all $n \geq m$. Then $Q \in P_n$ if and only if $n = m$ and we see that the collection $\{P_m : m \geq 1\}$ covers the collection

$$\{Q : \langle |f| \rangle_Q > 0 \text{ or } \langle |g| \rangle_Q > 0, \text{ and } Q \cap \Omega^{\complement} \neq \emptyset\}.$$

If $Q \in P_m$, then $Q \notin \tilde{P}_{m-1}$ so that if $R \supset Q$, then

$$\langle |f| \rangle_R, \langle |g| \rangle_R \lesssim 2^{-m}|E|^{-1}.$$

Let $\mathcal{X}_m$ denote the collection of maximal cubes in P_m. Now we continue decomposing the sum from where we left it

$$\begin{aligned}
&\sum_{\substack{Q \in \mathcal{D} \\ Q \cap \Omega^{\complement} \neq \emptyset}} |\alpha_Q| \langle |f| \rangle_Q \langle |g| \rangle_Q |\langle \psi, h_Q \rangle| \\
&\leq \sum_{m \geq 1} \sum_{Q_{MAX} \in \mathcal{X}_m} \sum_{\substack{Q \in P_m \\ Q \subset Q_{MAX}}} |\alpha_Q| \langle |f| \rangle_Q \langle |g| \rangle_Q |\langle \psi, h_Q \rangle| \\
&\leq \sum_{m \geq 1} \sum_{Q_{MAX} \in \mathcal{X}_m} \sum_{\substack{Q \in P_m \\ Q \subset Q_{MAX}}} |\alpha_Q| 2^{-m}|E|^{-1} 2^{-m}|E|^{-1} |\langle \psi, h_Q \rangle| \\
&\leq \sum_{m \geq 1} 2^{-2m}|E|^{-2} \sum_{Q_{MAX} \in \mathcal{X}_m} \sum_{\substack{Q \in P_m \\ Q \subset Q_{MAX}}} |\alpha_Q| |\langle \psi, h_Q \rangle| \\
&\leq \sum_{m \geq 1} 2^{-2m}|E|^{-2} \sum_{Q_{MAX} \in \mathcal{X}_m} \Big(\sum_{\substack{Q \in P_m \\ Q \subset Q_{MAX}}} |\alpha_Q|^2 \Big)^{1/2} \Big(\sum_{\substack{Q \in P_m \\ Q \subset Q_{MAX}}} |\langle \psi, h_Q \rangle|^2 \Big)^{1/2}.
\end{aligned}$$

To continue, we derive some intermediate estimates:

$$\Big(\sum_{\substack{Q \in P_m \\ Q \subset Q_{MAX}}} |\alpha_Q|^2 \Big)^{1/2} \leq |Q_{MAX}|^{1/2} \|\alpha\|_{BMO(2)},$$

and by the square function L^2- estimate (equality)

$$\Big(\sum_{\substack{Q \in P_m \\ Q \subset Q_{MAX}}} |\langle \psi, h_Q \rangle|^2 \Big)^{1/2} \leq \Big(\sum_{Q} |\langle 1_{Q_{MAX}} \psi, h_Q \rangle|^2 \Big)^{1/2} = \|1_{Q_{MAX}} \psi\|_{L^2} \leq |Q_{MAX}|^{1/2}.$$

Also as

$$\bigcup_{Q \in P_m} Q \subset \{\mathcal{M}_{\mathcal{D}} f > 2^{-m} C|E|^{-1}\} \cup \{\mathcal{M}_{\mathcal{D}} g > 2^{-m} C|E|^{-1}\}$$

and $\|f\|_{L^1} = \|g\|_{L^1} = 1$, it follows that

$$|\bigcup_{Q\in P_m} Q| \leq |\{\mathcal{M}_{\mathcal{D}} f > 2^{-m}C|E|^{-1}\}| + |\{\mathcal{M}_{\mathcal{D}} g > 2^{-m}C|E|^{-1}\}| \lesssim 2^m|E|.$$

Using these, the estimate continues as

$$\begin{aligned}
&\sum_{m\geq 1} 2^{-2m}|E|^{-2} \sum_{Q_{MAX}\in\mathcal{X}_m} (\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} |\alpha_Q|^2)^{1/2} (\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} |\langle\psi, h_Q\rangle|^2)^{1/2} \\
&\lesssim \sum_{m\geq 1} 2^{-2m}|E|^{-2} \sum_{Q_{MAX}\in\mathcal{X}_m} |Q_{MAX}|^{1/2}|Q_{MAX}|^{1/2} \\
&= \sum_{m\geq 1} 2^{-2m}|E|^{-2}|P_m| \\
&\lesssim \sum_{m\geq 1} 2^{-2m}|E|^{-2}2^m|E| \leq |E|^{-1}.
\end{aligned}$$

It remains to prove the case of a paraproduct of form 2. Assume that

$$\Pi_\alpha(f,g) = \sum_{Q\in\mathcal{D}} \alpha_Q \langle f\rangle_Q \langle g, h_Q\rangle \frac{1_Q}{|Q|}.$$

We use the same set E' as in the case 1. Since only cubes Q for which $\langle|f|\rangle_Q > 0$ convey any mass to the sums, the following splitting of the sum is justified. First, proceed as in the case 1. up to the point

$$\begin{aligned}
|\langle\Pi_\alpha(f,g),\psi\rangle| &\leq \sum_{\substack{Q\in\mathcal{D}\\ Q\cap\Omega^{\complement}\neq\emptyset}} |\alpha_Q|\langle|f|\rangle_Q|\langle g, h_Q\rangle|\langle|\psi|\rangle_Q \\
&\leq \sum_{m\geq 1}\sum_{Q_{MAX}\in\mathcal{X}_m}\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} |\alpha_Q|\langle|f|\rangle_Q|\langle g, h_Q\rangle|\langle|\psi|\rangle_Q \\
&\lesssim |E|^{-1}\sum_{m\geq 1} 2^{-m} \sum_{Q_{MAX}\in\mathcal{X}_m}\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} |\alpha_Q||\langle g, h_Q\rangle| \\
&\leq |E|^{-1}\sum_{m\geq 1} 2^{-m} \sum_{Q_{MAX}\in\mathcal{X}_m} (\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} |\alpha_Q|^2)^{1/2} (\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} |\langle g, h_Q\rangle|^2)^{1/2} \\
&\lesssim |E|^{-1}\sum_{m\geq 1} 2^{-m} \sum_{Q_{MAX}\in\mathcal{X}_m} |Q_{MAX}|^{1/2} (\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} |\langle g, h_Q\rangle|^2)^{1/2}.
\end{aligned}$$

To estimate the innermost sum, let $P_m g = \sum_{Q\in P_m}\Delta_Q g$. By lemma 1.41 and orthogonality of Haar functions we may write

$$\langle g, h_Q\rangle = \langle P_m g, h_Q\rangle.$$

Using this we first estimate the right-most term to

$$(\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} |\langle g,h_Q\rangle|^2)^{1/2} \leq (\sum_{Q\in\mathcal{D}} |\langle 1_{Q_{MAX}} P_m g, h_Q\rangle|^2)^{1/2}$$
$$= \|\mathcal{S}_{\mathcal{D}}(1_{Q_{MAX}} P_m g)\|_{L^2} = \|1_{Q_{MAX}} P_m g\|_{L^2} \leq \|P_m g\|_{L^\infty} |Q_{MAX}|^{1/2}.$$

Next we derive an estimate for $\|P_m g\|_{L^\infty}$.

Let $\mathcal{F}_k$ consist of maximal elements of $\mathcal{F}_k(f)\cup\mathcal{F}_k(g)$. We may assume that $x \in \operatorname{spt}(P_m g)$. We have two cases.

Case 1: There exists a cube $R(x)\in\mathcal{F}_{m-1}$ containing the point x. Then at the point x, $P_m g$ as a martingale difference telescopes to

$$P_m g(x) = \langle |g|\rangle_{R(x)} - \langle |g|\rangle_{Q(x)},$$

where $Q(x)\in\mathcal{F}_m$ and $R(x)\in\mathcal{F}_{m-1}$. Thus by the maximality of $Q(x)$ and $R(x)$ in these collections, respectively, it holds that

$$\langle |g|\rangle_{Q(x)} \lesssim_n \langle |g|\rangle_{Q(x)^{(1)}} \lesssim 2^{-m}|E|^{-1}, \qquad \langle |g|\rangle_{R(x)} \lesssim_n \langle |g|\rangle_{R(x)^{(1)}} \lesssim 2^{-(m-1)}|E|^{-1},$$

so that up to a dimensional constant we can estimate the value at the point x to

$$|P_m g(x)| = |\langle |g|\rangle_{Q(x)} - \langle |g|\rangle_{R(x)}| \leq \langle |g|\rangle_{Q(x)} + \langle |g|\rangle_{R(x)} \lesssim_n 2^{-m}|E|^{-1}.$$

Case 2: Let N denote the set where no such cube as in case 1 exists. Then by the Lebesgue differentiation theorem

$$|P_m g(x)| = |g(x) - \langle g\rangle_{Q(x)}| \lesssim_n 2^{-m}|E|^{-1} \text{ a.e. } x\in N.$$

Having these two cases together gives

$$\|P_m g\|_{L^\infty} \lesssim 2^{-m}|E|^{-1}.$$

By this, continuing the main estimate to an identical form as that in the first case, and likewise, by proceeding identically as in the first case we acquire

$$\begin{aligned}
&|E|^{-1}\sum_{m\geq 1} 2^{-m} \sum_{Q_{MAX}\in\mathcal{X}_m} |Q_{MAX}|^{1/2} (\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} |\langle g,h_Q\rangle|^2)^{1/2}\\
\lesssim&|E|^{-1}\sum_{m\geq 1} 2^{-m} \sum_{Q_{MAX}\in\mathcal{X}_m} |Q_{MAX}|^{1/2}\|P_m g\|_{L^\infty}|Q_{MAX}|^{1/2}\\
\lesssim&|E|^{-2}\sum_{m\geq 1} 2^{-2m} \sum_{Q_{MAX}\in\mathcal{X}_m} |Q_{MAX}|\\
=&|E|^{-2}\sum_{m\geq 1} 2^{-2m}|P_m| \lesssim |E|^{-1}.
\end{aligned}$$

□

Next, the same end-point estimate will be proved for bilinear shifts. The proof will be roughly of the same form as the previous one for bilinear paraproducts, but with the addition that the complexity (i, j, k) makes it slightly harder.

3.6 A weak end-point estimate for shifts

Theorem 3.7. *Assume that $f, g \in L^1$. Then*

$$\|S^{i,j,k}(f,g)\|_{L^{1/2,\infty}} \lesssim (1+\max(i,j,k))^2 \|f\|_{L^1}\|g\|_{L^1}.$$

Proof. We begin by first deriving intermediate estimates for shifts of two forms, the third being similar to the second, we skip it.

Assume first that the shift $S^{i,j,k}$ is of the form

$$S^{i,j,k}(f,g) = \sum_{Q\in\mathcal{D}} \sum_{I^i=J^j=K^k=Q} \alpha_{IJKQ}\langle f, h_I\rangle\langle g, h_J\rangle h_K^0.$$

By scaling we may assume that $\|f\|_{L^1} = \|g\|_{L^1} = 1$. Lemma 3.2 furnished with $\Omega, E', \psi, \mathcal{F}(f), \mathcal{F}(g), \tilde{P}_m, P_m$ which are as in the proof of theorem 3.5 will be used. Proceeding as there, checking the conditions of lemma 3.2, the condition $|E'| > |E|/2$ being immediate, the main work lies in estimating $|\langle S^{i,j,k}(f,g), \psi\rangle|$.

First, we estimate it upwards as

$$\begin{aligned}|\langle S^{i,j,k}(f,g),\psi\rangle| &= \Big|\langle \sum_{Q\in\mathcal{D}} \sum_{I^i=J^j=K^k=Q} \alpha_{IJKQ}\langle f,h_I\rangle\langle g,h_J\rangle\langle h_K^0,\psi\rangle\Big| \\ &\le \sum_{m\ge1}\sum_{Q_{MAX}\in\mathcal{X}_m}\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}}\sum_{\substack{I,J,K\subset Q\\ I^i=J^j=K^k=Q}} |\alpha_{IJKQ}||\langle f,h_I\rangle||\langle g,h_J\rangle||\langle\psi,h_K^0\rangle|.\end{aligned}$$

Recall that $\mathcal{X}_m$ is the subcollection of maximal cubes of P_m. Then, estimating the two innermost sums as in the proof of theorem 2.11 we get

$$
\begin{aligned}
&\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} \sum_{I^i=J^j=K^k=Q} |\alpha_{IJKQ}||\langle f,h_I\rangle||\langle g,h_J\rangle||\langle \psi,h_K^0\rangle| \\
\leq &\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} \sum_{I^i=J^j=K^k=Q} |\alpha_{IJKQ}|\langle|\Delta_Q^i f|,|h_I|\rangle\langle|\Delta_Q^j g|,|h_J|\rangle\langle|\psi|,|h_K^0|\rangle \\
\leq &\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} \sum_{I^i=J^j=K^k=Q} |Q|^{-2}\int_I |\Delta_Q^i f|\int_J |\Delta_Q^j g|\int_K |\psi| \\
= &\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} |Q|\langle|\Delta_Q^i f|\rangle_Q\langle|\Delta_Q^j g|\rangle_Q \\
\leq &\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} |Q|\langle|\Delta_Q^i f|\rangle_Q\langle|\Delta_Q^j g|\rangle_Q = \sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} \sqrt{|Q|}\langle|\Delta_Q^i f|\rangle_Q\sqrt{|Q|}\langle|\Delta_Q^j g|\rangle_Q \\
\leq &\Big(\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} |Q|\langle|\Delta_Q^i f|\rangle_Q^2\Big)^{1/2}\Big(\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} |Q|\langle|\Delta_Q^j g|\rangle_Q^2\Big)^{1/2},
\end{aligned}
$$

where

$$\Delta_Q^i f = \sum_{I^i=Q}\Delta_I f, \qquad \Delta_Q^j f = \sum_{J^j=Q}\Delta_I g.$$

Assume for the second intermediate estimate that a shift is of the form

$$S^{i,j,k}(f,g) = \sum_{Q\in\mathcal{D}}\sum_{I^i=J^j=K^k=Q}\alpha_{IJKQ}\langle f,h_I\rangle\langle g,h_J^0\rangle h_K.$$

First we estimate it upwards as

$$|\langle S^{i,j,k}(f,g),\psi\rangle| \leq \sum_{m\geq 1}\sum_{Q_{MAX}\in P_M}\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}}\sum_{I^i=J^j=K^k=Q}|\alpha_{IJKQ}||\langle f,h_I\rangle||\langle g,h_J^0\rangle||\langle\psi,h_K\rangle|.$$

Then we estimate the two innermost sums as

$$
\begin{aligned}
&\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} \sum_{I^i=J^j=K^k=Q} |\alpha_{IJKQ}||\langle f,h_I\rangle||\langle g,h_J^0\rangle||\langle \psi,h_K\rangle| \\
\leq &\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} \sum_{I^i=J^j=K^k=Q} \frac{\sqrt{|I||J||K|}}{|Q|^2}\langle|\Delta_Q^i f|,|h_I|\rangle\langle|g|,|h_J^0|\rangle\langle|\Delta_Q^k\psi|,|h_K|\rangle| \\
\leq &\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}} |Q|\langle|\Delta_Q^i f|\rangle_Q\langle|g|\rangle_Q\langle|\Delta_Q^k\psi|\rangle_Q
\end{aligned}
$$

$$\lesssim 2^{-m}|E|^{-1}\Big(\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}}|Q|\langle|\Delta_Q^i f|\rangle_Q^2)^{1/2}\Big(\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}}|Q|\langle|\Delta_Q^k \psi|\rangle_Q^2)^{1/2}$$
$$\lesssim 2^{-m}|E|^{-1}\Big(\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}}|Q|\langle|\Delta_Q^i f|\rangle_Q^2)^{1/2}|Q_{MAX}|^{1/2},$$

where the last estimate derives by lemma 2.5 and the square function estimate, theorem 1.32 as

$$\begin{aligned}(\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}}|Q|\langle|\Delta_Q^k \psi|\rangle_Q^2)^{1/2} &\leq (\sum_{Q\in\mathcal{D}}|Q|\langle|\Delta_Q^k 1_{Q_{MAX}}\psi|^2\rangle_Q)^{1/2} = (\sum_{Q\in\mathcal{D}}\int_Q|\Delta_Q^k 1_{Q_{MAX}}\psi|^2)^{1/2}\\
&\leq (\sum_{Q\in\mathcal{D}}\sum_{I^i=Q}\int_Q|\Delta_I 1_{Q_{MAX}}\psi|^2)^{1/2} = (\int_Q\sum_{Q\in\mathcal{D}}|\Delta_Q 1_{Q_{MAX}}\psi|^2)^{1/2}\\
&= \|\mathcal{S}_{\mathcal{D}}1_{Q_{MAX}}\psi\|_{L^2} = \|1_{Q_{MAX}}\psi\|_{L^2}\leq |Q_{MAX}|^{1/2}.\end{aligned}$$

Now, if we could prove the following estimate

$$(\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}}|Q|(\langle|\Delta_Q^i\phi|\rangle_Q)^2)^{1/2}\lesssim(1+i)2^{-m}|E|^{-1}|Q_{MAX}|^{1/2}$$

for $\phi = f, g$, then applying this, the rest of the proof would in both cases verbatim be the same as the last estimate in the proof of theorem 3.5. Nevertheless, we repeat the estimate in the first case.

As assumptions placed on f and g are identical, it is enough to prove the estimate for $\phi = f$.

Proof of the remaining estimate: Let $f = g + b$ be the CZD of the function f at the level $\lambda = 2^{-(m-1)}|E|^{-1}C$. Notice that this function g is not the same function g that appears before in this proof. First we estimate

$$\begin{aligned}&(\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}}|Q|\langle|\Delta_Q^i f|\rangle_Q^2)^{1/2}\leq(\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}}|Q|\langle|\Delta_Q^i g|\rangle_Q^2)^{1/2}+(\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}}|Q|\langle|\Delta_Q^i b|\rangle_Q^2)^{1/2}\\
=&\|(\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}}\langle|\Delta_Q^i g1_Q|\rangle_Q^2 1_Q)^{1/2}\|_{L^2}+\|(\sum_{\substack{Q\in P_m\\ Q\subset Q_{MAX}}}\langle|\Delta_Q^i b|\rangle_Q^2 1_Q)^{1/2}\|_{L^2} = I + II.\end{aligned}$$

By the CZD of the function f we have that $\|g\|_{L^\infty}\lesssim 2^{-m}|E|^{-1}$ and so I can be estimated upwards to

$$I\leq\|(\sum_{Q\in\mathcal{D}}\langle|\Delta_Q^i g1_{Q_{MAX}}|\rangle_Q^2)^{1/2}1_{Q_{MAX}}\|_{L^2}\lesssim\|g1_{Q_{MAX}}\|_{L^2}\lesssim 2^{-m}|E|^{-1}|Q_{MAX}|^{1/2},$$

which is of the form we want. It remains to estimate II. First recall that

$$\mathcal{F}_m(f) = \{Q_{MAX} : Q_{MAX}\text{ is maximal on the condition }\langle|f|\rangle_{Q_{MAX}} > 2^{-m}C|E|^{-1}\}.$$

Thus

$$b = \sum_{Q \in \mathcal{F}_{m-1}(f)} b_Q.$$

By the John-Nirenberg inequality, theorem 2.16, we get

$$\begin{aligned}
II &= |Q_{MAX}|^{1/2} \frac{1}{|Q_{MAX}|^{1/2}} \|(\sum_{\substack{Q \in P_m \\ Q \subset Q_{MAX}}} \langle |\Delta_Q^i b| \rangle_Q^2 1_Q)^{1/2} \|_{L^2} \\
&\leq |Q_{MAX}|^{1/2} \sup_{Q_0 \in P_m} \frac{1}{|Q_0|^{1/2}} \|(\sum_{\substack{Q \in P_m \\ Q \subset Q_0}} \langle |\Delta_Q^i b| \rangle_Q^2 1_Q)^{1/2} \|_{L^2} \\
&\sim |Q_{MAX}|^{1/2} \sup_{Q_0 \in P_m} \frac{1}{|Q_0|} \|(\sum_{\substack{Q \in P_m \\ Q \subset Q_0}} \langle |\Delta_Q^i b| \rangle_Q^2 1_Q)^{1/2} \|_{L^1}.
\end{aligned}$$

Let $Q_0 \in P_m$ be arbitrary. Then

$$\begin{aligned}
&\|(\sum_{\substack{Q \in P_m \\ Q \subset Q_0}} \langle |\Delta_Q^i b| \rangle_Q^2 1_Q)^{1/2} \|_{L^1} \leq \| \sum_{\substack{Q \in P_m \\ Q \subset Q_0}} \langle |\Delta_Q^i b| \rangle_Q 1_Q \|_{L^1} \leq \sum_{\substack{Q \in P_m \\ Q \subset Q_0}} \|\Delta_Q^i b 1_Q \|_{L^1} \\
&\leq \sum_{\substack{Q \in P_m \\ Q \subset Q_0}} \sum_{I^i = Q} \int_Q |\Delta_I^i b| \leq \sum_{\eta} \sum_{\substack{Q \in P_m \\ Q \subset Q_0}} \sum_{I^i = Q} |\langle b, h_I^\eta \rangle| |I|^{1/2}.
\end{aligned}$$

We split the sum further. Assume that $J \in \mathcal{F}_{m-1}$. If $J \subsetneq I$, then

$$\langle b_J, h_I^\eta \rangle = \langle h_I^\eta \rangle_J \int b_J = 0.$$

As $Q \notin \tilde{P}_{m-1}$, we cannot have $Q \subset J$. Thus we may continue the estimate as

$$\begin{aligned}
&\sum_{\eta} \sum_{\substack{Q \in P_m \\ Q \subset Q_0}} \sum_{I^i = Q} |\langle b, h_I^\eta \rangle| |I|^{1/2} \leq \sum_{\eta} \sum_{\substack{Q \in P_m \\ Q \subset Q_0}} \sum_{I^i = Q} \sum_{\substack{J \in \mathcal{F}_{m-1}(f) \\ I \subset J \subsetneq Q}} |\langle b_J, h_I^\eta \rangle| |I|^{1/2} \\
&\leq \sum_{\eta} \sum_{\substack{Q \in P_m \\ Q \subset Q_0}} \sum_{I^i = Q} \sum_{\substack{J \in \mathcal{F}_{m-1}(f) \\ I \subset J \subsetneq Q}} \int_I |b_J| = \sum_{\eta} \sum_{l=0}^{i-1} \sum_{\substack{Q \in P_m \\ Q \subset Q_0}} \sum_{\substack{J \in \mathcal{F}_{m-1}(f) \\ J^{i-l} = Q}} \sum_{I^l = J} \int_I |b_J| \\
&= \sum_{\eta} \sum_{l=0}^{i-1} \sum_{\substack{Q \in P_m \\ Q \subset Q_0}} \sum_{\substack{J \in \mathcal{F}_{m-1}(f) \\ J^{i-l} = Q}} \int_J |b_J| \overset{*}{\leq} 2 \sum_{\eta} \sum_{l=0}^{i-1} \sum_{\substack{Q \in P_m \\ Q \subset Q_0}} \sum_{\substack{J \in \mathcal{F}_{m-1}(f) \\ J^{i-l} = Q}} \langle |f| \rangle_J |J| \\
&\leq 2^{-m} |E|^{-1} \sum_{\eta} \sum_{l=0}^{i-1} \sum_{\substack{Q \in P_m \\ Q \subset Q_0}} \sum_{\substack{J \in \mathcal{F}_{m-1}(f) \\ J^{i-l} = Q}} |J|
\end{aligned}$$

$$
\begin{aligned}
&\overset{**}{\lesssim} 2^{-m}|E|^{-1}\sum_{\eta}\sum_{l=0}^{i-1}|Q_0| = 2^{-(m-1)}|E|^{-1}(2^n-1)i|Q_0| \\
&\lesssim i2^{-m}|E|^{-1}|Q_0|.
\end{aligned}
$$

The passing at $*$ uses the fact that $\|b_J\|_{L^1} \leq 2\|f\|_{L^1}$, and the passing at $**$ follows from the disjointness of cubes in the collection $\mathcal{F}_{m-1}(f)$. By this we get

$$
(\sum_{\substack{Q\in P_m \\ Q\subset Q_{MAX}}} |Q|\langle|\Delta_Q^i f|\rangle_Q^2)^{1/2} \leq I + II \lesssim (1+i)2^{-m}|E|^{-1}|Q_{MAX}|^{1/2},
$$

and the identical estimate for the term with g gives

$$
(\sum_{\substack{Q\in P_m \\ Q\subset Q_{MAX}}} |Q|\langle|\Delta_Q^j g|\rangle_Q^2)^{1/2} \lesssim (1+j)2^{-m}|E|^{-1}|Q_{MAX}|^{1/2}.
$$

Combining these, we have the claim in the first case.

$$
\begin{aligned}
\left|\langle S^{i,j,k}(f,g),\psi\rangle\right| &\lesssim \sum_{m\geq 1}\sum_{Q_{MAX}\in\mathcal{X}_m}(1+i)2^{-m}|E|^{-1}|Q_{MAX}|^{1/2}(1+j)2^{-m}|E|^{-1}|Q_{MAX}|^{1/2} \\
&\leq (1+\max(i,j,k))^2\sum_{m\geq 1}2^{-2m}|E|^{-2}|P_m| \\
&\lesssim (1+\max(i,j,k))^2|E|^{-1}.
\end{aligned}
$$

Similarly we get the claim in the second case. □

This concludes chapter 3. The goal of the next chapter is to go through an interpolation technique that allows us to extend the results of this and the previous chapter and prove

Theorem 3.8. *Assume that Λ is either a bilinear shift or a bilinear paraproduct. Then for $1 < p, q < \infty$ and $r > 1/2$ satisfying $1/p + 1/q = 1/r$,*

$$
\|\Lambda(f,g)\|_{L^r} \lesssim \|f\|_{L^p}\|g\|_{L^q}.
$$

4 Interpolation of bilinear operators

The setting of this chapter will consider bilinear operators. With suitable modifications the maint result of this chapter, theorem 4.8, runs through in the n-linear setting similarly as it does here.

In the previous chapters estimates for bilinear shifts and bilinear paraproducts were proven in two different situations: the weak end-point estimates in the quasi-Banach range and the strong estimates in the Banach range. The aim of this chapter is to go through a method of interpolation that allows us to conclude strong estimates in the quasi-Banach range from the weak end-point estimates in the quasi-Banach range and the strong estimates in the Banach range.

For another treatment of the interpolation in this chapter see [4] or [6].

Definition 4.1. *Assume that $0 < p, q < \infty$ and $0 < r < \infty$, and denote $\alpha = (p, q, r)$.*

A bilinear operator Λ satisfies a restricted weak-type (r.w.t.) estimate with(or at the point) α, if for all sets E_i such that $0 < |E_i| < \infty$ there is a major subset E of E_3 such that $|E| \geq \frac{|E_3|}{2}$ so that for all functions f_i for which it holds that

$$|f_i| \leq 1_{E_i} \text{ for } i = 1, 2 \text{ and } |f_3| \leq 1_{E'},$$

the following estimate is satisfied

$$|\langle \Lambda(f_1, f_2), f_3 \rangle| \leq C|E_1|^{1/p}|E_2|^{1/q}|E_3|^{1/r'}.$$

Remark 4.2. The points p, q, r are not linked by the usual relation $1/p + 1/q = 1/r$.

Remark 4.3. The points p, q, r being as in the definition, $1/p, 1/q \in (0, \infty)$ and $1/r' \in (-\infty, 1)$.

Remark 4.4. Let $0 < r < \infty$ and assume that Λ satisfies the r.w.t. estimate at a point (p, q, r). By lemma 3.2 this is equivalent with the statement: For all f_i such that $|f_i| \leq 1_{E_i}$, where $0 < |E_i| < \infty$, for $i = 1, 2$, it holds that

$$\|\Lambda(f_1, f_2)\|_{L^{r,\infty}} \lesssim |E_1|^{1/p}|E_2|^{1/q}.$$

Especially if an operator $\Lambda : L^q \times L^p \longrightarrow L^{r,\infty}$ is bounded, it satisfies the r.w.t. estimate at the point (p, q, r).

In the case $r > 1$ we may let go of the major subset condition, see remark 4.6. For this

Lemma 4.5. *Let $f \in L^0$, $0 < p < q < \infty$ and $0 < |E| < \infty$. Then*

$$\langle |f|^p \rangle_E^{1/p} \lesssim_{p,q} \frac{\|1_E f\|_{L^{q,\infty}}}{|E|^{1/q}}.$$

Proof. By Cavalieri's principle

$$\begin{aligned}\int_E |f|^p &\sim_p \int_0^\infty \lambda^{p-1}|\{|1_E f| > \lambda\}|\, d\lambda \\ &= \int_0^A \lambda^{p-1}|\{|1_E f| > \lambda\}|\, d\lambda + \int_A^\infty \lambda^{p-1}|\{|1_E f| > \lambda\}|\, d\lambda \\ &=: I + II.\end{aligned}$$

Choosing $A > 0$ suitably as $A = \|1_E f\|_{L^{q,\infty}}|E|^{-1/q}$ one may estimate the integrals I and II.

Integral I:

$$I \lesssim_p |E| \int_0^A \lambda^{p-1}\, d\lambda \sim_p |E|\Big(\|1_E f\|_{L^{q,\infty}}|E|^{-1/q}\Big)^p = |E|\frac{\|1_E f\|^p_{L^{q,\infty}}}{|E|^{p/q}}.$$

Since $\|f\|_{L^{q,\infty}} = \sup_{t>0} t|\{|f| > t\}|^{1/q}$ and $p-q-1 < -1$ we may estimate the integral II as

$$\begin{aligned}II &= \int_A^\infty \lambda^{p-1}|\{|1_E f| > \lambda\}|\, d\lambda \leq \int_A^\infty \lambda^{p-1}\Big(\frac{\|1_E f\|_{L^{q,\infty}}}{\lambda}\Big)^q\, d\lambda \\ &= \|1_E f\|^q_{L^{q,\infty}} \int_A^\infty \lambda^{p-q-1}\, d\lambda \\ &\lesssim_{p,q} \|1_E f\|^q_{L^{q,\infty}}\Big(\|1_E f\|_{L^{q,\infty}}|E|^{-1/q}\Big)^{p-q} \\ &= |E|\frac{\|1_E f\|^p_{L^{q,\infty}}}{|E|^{p/q}}.\end{aligned}$$

Combining estimates I and II gives

$$\int_E |f|^p = I + II \lesssim_{p,q} |E|\frac{\|1_E f\|^p_{L^{q,\infty}}}{|E|^{p/q}}.$$

Dividing by $|E|$ and taking the p:th root gives the claim. □

Remark 4.6. In the case $r > 1$ we may let go of the major subset condition:

Assume that Λ satisfies the r.w.t. estimate at the point (p, q, r). Then for f_i such that $|f_i| \leq 1_{E_i}$, $i = 1, 2$, by lemma 3.2 it holds that

$$\|\Lambda(f_1, f_2)\|_{L^{r,\infty}} \lesssim |E_1|^{1/p}|E_2|^{1/q}.$$

By lemma 4.5 for f_3 such that $|f_3| \leq 1_{E_3}$ we can then estimate

$$|\langle\Lambda(f_1, f_2), f_3\rangle| \leq \langle|\Lambda(f_1, f_2)|, 1_{E_3}\rangle \lesssim_r \|\Lambda(f_1, f_2)\|_{L^{r,\infty}}|E_3|^{1/r'} \lesssim |E_1|^{1/p}|E_2|^{1/q}|E_3|^{1/r'}.$$

The following lemma will be used repeatedly in the proof of theorem 4.8.

Lemma 4.7. *Assume that* $f \in L^{r,\infty}$ *for some* $r > 0$ *and let* $\Omega = \{f \neq 0\}$. *Then exists a collection of disjoint sets* $\{F_k\}_{k\in\mathbb{Z}}$ *such that* $|F_k| = 2^k$,

$$\Omega \subset \bigcup_{k\in\mathbb{Z}} F_k,$$

and

$$\operatorname*{essinf}_{F_k}(|f|) \geq \operatorname*{esssup}_{F_{k+1}}(|f|).$$

Especially we have a representation of the function f *as*

$$f = \sum_{k\in\mathbb{Z}} f 1_{F_k}.$$

Here we have defined

$$\operatorname*{esssup}_{F_k}(f) = \inf\{c \in \mathbb{R} : |\{f > c\} \cap F_k| = 0\}, \qquad \operatorname*{essinf}_{F_k}(f) = \sup\{c \in \mathbb{R} : |\{f < c\} \cap F_k| = 0\}.$$

Proof. Let $\gamma > 0$ be arbitrary and define

$$\lambda = \sup\{\alpha : |\{|f| \geq \alpha\}| \geq \gamma\}.$$

As $f \in L^{r,\infty}$, λ is finite. Then for all $\lambda' > \lambda$ and for all $\lambda'' < \lambda$ it holds that

$$|\{|f| > \lambda'\}| < \gamma \qquad |\{|f| \geq \lambda''\}| \geq \gamma$$

so that by convergence theorems of measure acquire

$$|\{|f| > \lambda\}| \leq \gamma, \qquad \{|f| \geq \lambda\} \geq \gamma.$$

By this choose the maximal radius $r \in [0, \infty]$ so that

$$\Big|\{|f| > \lambda\} \cup \Big(\{|f| = \lambda\} \cap B(0, r)\Big)\Big| = \gamma.$$

Define

$$A_\gamma = \{|f| > \lambda\} \cup \Big(\{|f| = \lambda\} \cap B(0, r)\Big).$$

Then by construction $|A_\gamma| = \gamma$, and if $\gamma'' < \gamma' < \gamma$, then

$$A_{\gamma''} \subset A_{\gamma'} \subset A_\gamma.$$

Moreover, if we let $B = A_{\gamma'} \setminus A_{\gamma''}$ and $C = A_\gamma \setminus A_{\gamma'}$, then

$$\operatorname*{essinf}_{B}(f) \geq \operatorname*{esssup}_{C}(f).$$

Let $\gamma_k = 2^{k+1}$. For all $k \in \mathbb{Z}$, by the previous method we find a nested sequence of sets

$$\cdots \subset A_{\gamma_{k-1}} \subset A_{\gamma_k} \subset A_{\gamma_{k+1}} \subset \dots$$

such that $|A_{\gamma_k}| = \gamma_k$. Now we define $F_k = A_{\gamma_k} \setminus A_{\gamma_{k-1}}$. Then

$$|F_k| = |A_{\gamma_k}| - |A_{\gamma_{k-1}}| = \gamma_k - \gamma_{k-1} = 2^k.$$

Also, by construction

$$\operatorname*{essinf}_{F_{k-1}}(|f|) \geq \operatorname*{esssup}_{F_k}(|f|).$$

It remains to notice that $\Omega \subset \bigcup_{k\in\mathbb{Z}} F_k$. Indeed, since $f \in L^{r,\infty}$, it holds that

$$\Big|\{|f| > \alpha\}\Big| < \infty, \text{ for all } \alpha > 0.$$

From the construction it is then evident that $\{|f| > \alpha\} \subset \bigcup_{k\in\mathbb{Z}} A_{\gamma_k}$ and thus

$$\Omega = \bigcup_{\alpha>0} \{|f| > \alpha\} \subset \bigcup_{k\in\mathbb{Z}} A_{\gamma_k} = \bigcup_{k\in\mathbb{Z}} F_k.$$

$\square$

For the statement of the next theorem, we define $(p_{\varepsilon_1}, q_{\varepsilon_2}, r_{\varepsilon_3})$ to be the point for which

$$(1/p_{\varepsilon_1}, 1/q_{\varepsilon_2}, 1/r'_{\varepsilon_3}) = (1/p + \varepsilon_1, 1/q + \varepsilon_2, 1/r' + \varepsilon_3.)$$

Also, if we can decompose a function f as in lemma 4.7, we denote

$$f^N = \sum_{k=-N}^{N} f1_{F_k}.$$

Theorem 4.8. *Fix $\varepsilon > 0$ and let Λ be a bilinear operator satisfying the restricted weak-type estimate at the four points*

$$(p_{0-\varepsilon}, q_0, r_{0+\varepsilon}), \qquad (p_{0+\varepsilon}, q_0, r_{0-\varepsilon}), \qquad (p_0, q_{0-\varepsilon}, r_{0+\varepsilon}), \qquad (p_0, q_{0+\varepsilon}, r_{0-\varepsilon}),$$

where $p_0, q_0 \in (1, \infty)$ and $r_0 \in (1/2, 1]$ satisfy the relation $\frac{1}{p_0} + \frac{1}{q_0} = \frac{1}{r_0}$. Then

$$\|\Lambda(f, g)\|_{L^{r_0}} \lesssim_{p_0,q_0} M\|f\|_{L^{p_0}}\|g\|_{L^{q_0}},$$

where the constant M has dependence on the initial four points, i.e. on ε and the strength of the r.w.t. estimate at these points.

Proof. It is enough to prove the claim in a dense subset of $L^p \times L^q$ since then we can extend the boundedness of the operator Λ to the whole space. The dense subset is chosen as $\mathcal{F} \times \mathcal{F}$, where

$$\mathcal{F} = \Big\{ f^N : f \in L^\infty_c,\ N \in \mathbb{N} \Big\}.$$

So we assume that $f^N, g^M \in \mathcal{F}$, and we have their decompositions:

$$f^N = \sum_{k=-N}^{N} f_k, \qquad g^M = \sum_{k=-M}^{M} g_k.$$

Then as $r_0 \in (0,1]$

$$\|\Lambda(f^N, g^M)\|_{L^{r_0}}^{r_0} \overset{*}{=} \| \sum_{\substack{|i|\le N \\ |j|\le M}} \Lambda(f_i, g_j)\|_{L^{r_0}}^{r_0} \le \sum_{i,j} \|\Lambda(f_i, g_j)\|_{L^{r_0}}^{r_0}.$$

The passing at * is justified since the sum is finite and the last summation is only over indices i, j for which $|i| \le N$ and $|j| \le M$.

We momentarily fix indices i, j. As $f_i, g_j \in L^\infty_c$, by lemma 3.2, $|\Lambda(f_i, g_j)| \in L^{1,\infty}$. We may then again apply lemma 4.7 to acquire a collection $\{H_k\}$ corresponding to $|\Lambda(f_i, g_j)|$ and by this split and estimate further

$$\|\Lambda(f_i, g_j)\|_{L^{r_0}}^{r_0} = \int \Big| \sum_k \Lambda(f_i, g_j) 1_{H_k} \Big|^{r_0} \le \sum_k \int_{H_k} |\Lambda(f_i, g_j)|^{r_0}.$$

By Hölder's inequality we may estimate

$$\int_{H_k} |\Lambda(f_i, g_j)|^{r_0} \le \Big(\frac{1}{|H_k|} \int_{H_k} |\Lambda(f_i, g_j)| \Big)^{r_0} |H_k| = \Big(\int_{H_k} |\Lambda(f_i, g_j)| \Big)^{r_0} 2^{k(1-r_0)}.$$

Assume then that (p, q, r) is one of the four points at which by assumption the r.w.t. estimate is satisfied, and let $\tilde{H}_k = \bigcup_{l<k} H_l$. Then applying the r.w.t. estimate with the triple $(F_i, G_j, \tilde{H}_k \cup H_k)$ at this point we acquire a set $E \subset \tilde{H}_k \cup H_k$ such that

$$|E| \ge \frac{\tilde{H}_k \cup H_k}{2}.$$

By this and the relation given by lemma 4.7,

$$\operatorname*{essinf}_{\tilde{H}_k}(|\Lambda(f_i, g_j)|) \ge \operatorname*{esssup}_{H_k}(|\Lambda(f_i, g_j)|),$$

we have

$$\int_{H_k} |\Lambda(f_i, g_j)| \le \int_E |\Lambda(f_i, g_j)|.$$

From here we continue with the r.w.t. estimate to

$$\begin{aligned}\int_E |\Lambda(f_i,g_j)| =& \|f_i\|_{L^\infty}\|g_j\|_{L^\infty}\int \Lambda\Big(\frac{f_i}{\|f_i\|_{L^\infty}},\frac{g_j}{\|g_j\|_{L^\infty}}\Big)\frac{|\Lambda(f_i,g_j)|}{\Lambda(f_i,g_j)}1_E\\ \leq& M\|f_i\|_{L^\infty}\|g_j\|_{L^\infty}|F_i|^{1/p}|G_j|^{1/q}|\tilde{H}_k\cup H_k|^{1/r'}\\ \lesssim& M\|f_i\|_{L^\infty}\|g_j\|_{L^\infty}2^{i/p}2^{j/q}2^{k/r'}.\end{aligned}$$

To get a further estimate on $\int_{H_k}|\Lambda(f_i,g_j)|$, it is estimated in cases according to which of the following conditions hold

$\underline{1}.\ i-k\geq|j-k|,\qquad \underline{2}.\ k-i\geq|j-k|,\qquad \underline{3}.\ j-k\geq|i-k|,\qquad \underline{4}.\ k-j\geq|i-k|,$

Assume for example that 3. holds. Then as the r.w.t. estimate is satisfied at the point $(p_0,q_{0-\varepsilon},r_{0+\varepsilon})$ we may apply the previous estimate to acquire

$$\begin{aligned}\int_{H_k}|\Lambda(f_i,g_j)| &\lesssim_{p_0,q_0} M\|f_i\|_{L^\infty}\|g_j\|_{L^\infty}2^{i/p_0}2^{j(1-\varepsilon q_0)/q_0}2^{k(1+\varepsilon r_0')/r_0'}\\ &= M\|f_i\|_{L^\infty}\|g_j\|_{L^\infty}2^{i/p_0}2^{j/q_0}2^{k/r_0'}2^{-\varepsilon(j-k)}.\end{aligned}$$

Likewise, in the other cases by using the r.w.t. estimate assumption at the remaining three points we acquire similar estimates. Putting these four case-sensitive estimates together gives

$$\int_{H_k}|\Lambda(f_i,g_j)|\lesssim_{p_0,q_0} M\|f_i\|_{L^\infty}\|g_j\|_{L^\infty}2^{i/p_0}2^{j/q_0}2^{k/r_0'}2^{-\epsilon\max(|i-k|,|j-k|)}.$$

We have showed that

$$\|\Lambda(f^N,g^M)\|_{L^{r_0}}^{r_0}\lesssim_{p_0,q_0}\sum_{i,j}2^{k(1-r_0)}\sum_k\Big(M\|f_i\|_{L^\infty}\|g_j\|_{L^\infty}2^{i/p_0}2^{j/q_0}2^{k/r_0'}2^{-\epsilon\max(|i-k|,|j-k|)}\Big)^{r_0}.$$

Before proceeding to the last estimate, we go through some calculations for it to pass nicely:

A:

$$2^{k(1-r_0)}2^{r_0k/r_0'}=2^{k(1-r_0)+k(r_0-1)}=1,$$

B:

$$\begin{aligned}\sum_{k,j}2^{-r_0\epsilon\max(|i-k|,|j-k|)}&=\sum_k\sum_{\substack{j:\\|i-k|\geq|j-k|}}2^{-r_0\epsilon|i-k|}+\sum_k\sum_{\substack{j:\\|i-k|<|j-k|}}2^{-r_0\epsilon|j-k|}\\ &\lesssim_{p_0,q_0,\varepsilon}\sum_k|i-k|2^{-r_0\epsilon|i-k|}+\sum_k 2^{-r_0\epsilon|i-k|}\lesssim_{p_0,q_0,\varepsilon}1,\end{aligned}$$

C:

$$\big(\sum_i 2^i\|f_i\|_{L^\infty}^{p_0}\big)^{\frac{r_0}{p_0}}\lesssim\big(\sum_i|F_{i-1}|\operatorname*{essinf}_{F_{i-1}}(|f|)^{p_0}\big)^{\frac{r_0}{p_0}}\leq\|f\|_{L^{p_0}}^{r_0}.$$

Reference to the above A, B, C will be made by superscripts. Then

$$
\begin{aligned}
&\sum_k 2^{k(1-r_0)} \sum_{i,j} \Big(M \|f_i\|_{L^\infty} \|g_j\|_{L^\infty} 2^{i/p_0} 2^{j/q_0} |2|^{k/r_0'} 2^{-\epsilon \max(|i-k|,|j-k|)} \Big)^{r_0} \\
\overset{A}{=} & M^{r_0} \sum_k \sum_{i,j} \|f_i\|_{L^\infty}^{r_0} \|g_j\|_{L^\infty}^{r_0} 2^{r_0 i/p_0} 2^{r_0 j/q_0} 2^{-r_0 \epsilon \max(|i-k|,|j-k|)} \\
= & M^{r_0} \sum_{k,i,j} \Big(2^{\frac{r_0 i}{p_0}} \|f_i\|_{L^\infty}^{r_0} 2^{-r_0\epsilon \max(|i-k|,|j-k|) \frac{r_0}{p_0}} \Big) \Big(2^{\frac{r_0 j}{q_0}} \|g_j\|_{L^\infty}^{r_0} 2^{-r_0 \epsilon \max(|i-k|,|j-k|) \frac{r_0}{q_0}} \Big) \\
\leq & M^{r_0} \Big(\sum_{k,i,j} 2^i \|f_i\|_{L^\infty}^{p_0} 2^{-r_0 \epsilon \max(|i-k|,|j-k|)} \Big)^{\frac{r_0}{p_0}} \Big(\sum_{k,i,j} 2^j \|g_j\|_{L^\infty}^{q_0} 2^{-r_0 \epsilon \max(|i-k|,|j-k|)} \Big)^{\frac{r_0}{q_0}} \\
= & M^{r_0} \Big(\sum_i 2^i \|f_i\|_{L^\infty}^{p_0} \sum_{k,j} 2^{-r_0 \epsilon \max(|i-k|,|j-k|)} \Big)^{\frac{r_0}{p_0}} \\
& \times \Big(\sum_j 2^j \|g_j\|_{L^\infty}^{q_0} \sum_{k,i} 2^{-r_0 \epsilon \max(|i-k|,|j-k|)} \Big)^{\frac{r_0}{q_0}} \\
\overset{B}{\lesssim}_{p_0,q_0} & M^{r_0} \big(\sum_i 2^i \|f_i\|_{L^\infty}^{p_0} \big)^{\frac{r_0}{p_0}} \big(\sum_j 2^j \|g_j\|_{L^\infty}^{q_0} \big)^{\frac{r_0}{q_0}} \overset{C}{\lesssim} M^{r_0} \|f^N\|_{L^{p_0}}^{r_0} \|g^M\|_{L^{q_0}}^{r_0}.
\end{aligned}
$$

We have proved that

$$\|\Lambda(f^N, g^M)\|_{L^{r_0}} \lesssim_{p_0,q_0} M \|f^N\|_{L^{p_0}} \|g^M\|_{L^{q_0}}.$$

□

Remark 4.9. It is not necessary to have a uniform ε on the four points, and no restriction such as "$1/p + 1/q = 1/r$" is placed on these four points, even though these happen to satisfy it. This restriction is only placed on the center point (p_0, q_0, r_0). So, one can easily tune the assumptions of the theorem to get slightly different versions of it.

Anyways, this version of the theorem is more than enough, since we will next interpolate the restricted weak-type estimate to the whole range of $r \in (1/2, 1]$ and $p, q > 1$ satisfying the relation $1/p + 1/q = 1/r$.

Lemma 4.10. *Assume that $p_0, q_0 \in (1, \infty)$ and $r_0 \in (1/2, 1]$ satisfy the relation $1/p_0 + 1/q_0 = 1/r_0$. Then exist $1 < p, q, r < \infty$ and $\theta \in (0, 1)$ that satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and*

$$\underline{1.}: \ \frac{1-\theta}{1} + \frac{\theta}{p} = \frac{1}{p_0}, \qquad \underline{2.}: \ \frac{1-\theta}{1} + \frac{\theta}{q} = \frac{1}{q_0}, \qquad \underline{3.}: \ \frac{1-\theta}{1/2} + \frac{\theta}{r} = \frac{1}{r_0}.$$

Proof. We may assume that $p_0 \geq q_0$. We solve equation 1 with $p > p_0$ large enough, i.e. with p that satisfies

$$1/p + 1/q_0 < 1.$$

This determines θ, q, r, and shows that $\theta \in (0,1)$. By equation 2 since $\theta \in (0,1)$, it is necessarily the case that $q > q_0 > 1$. By substituting the left-hand sides of equations $1. - 3.$ to $1/p_0 + 1/q_0 = 1/r_0$ it follows that

$$1/p + 1/q = 1/r.$$

Since p was large enough and $q > q_0 > 1$, this shows that

$$1/r = 1/p + 1/q < 1/p + 1/q_0 < 1,$$

i.e. $r > 1$. $\square$

Lemma 4.11. *Assume that Λ is a bilinear operator satisfying the r.w.t. estimate at the point $(1,1,1/2)$ and at all points (p,q,r) that satisfy the relation $1/p+1/q = 1/r$ and are such that $1 < p,q,r < \infty$. Assume that $r_0 \in (1/2,1]$ and $1 < p_0, q_0 < \infty$. Then Λ satisfies the r.w.t. estimate at the point (p_0, q_0, r_0).*

Proof. By lemma 4.10 there exist $1 < p,q,r < \infty$ and $\theta \in (0,1)$ that satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and

$$\underline{1.}: \frac{1-\theta}{1} + \frac{\theta}{p} = \frac{1}{p_0}, \qquad \underline{2.}: \frac{1-\theta}{1} + \frac{\theta}{q} = \frac{1}{q_0}, \qquad \underline{3.}: \frac{1-\theta}{1/2} + \frac{\theta}{r} = \frac{1}{r_0}.$$

Here we acquire the set E in the situation of the r.w.t. estimate at the point $(1,1,1/2)$ and by remark 4.6 this set works also for the tuples $p,q,r > 1$.

Then by the r.w.t. assumption for all sets E_i such that $0 < |E_i| < \infty$, there is a major subset E of E_3 such that $|E| \geq \frac{|E_3|}{2}$ and for all functions f_i for which it holds that

$$|f_i| \leq 1_{E_i} \text{ for } i = 1,2 \text{ and } |f_3| \leq 1_{E'},$$

the estimates

$$|\langle \Lambda(f_1,f_2), f_3\rangle|^{1-\theta} \lesssim \left(|E_1|^1 |E_2|^1 |E_3|^{-1}\right)^{1-\theta}$$

and

$$|\langle \Lambda(f_1,f_2), f_3\rangle|^{\theta} \lesssim \left(|E_1|^{1/p} |E_2|^{1/q} |E_3|^{1/r'}\right)^{\theta}$$

hold. Multiplying these together gives

$$\begin{aligned}|\langle \Lambda(f_1,f_2), f_3\rangle| &\lesssim \left(|E_1|^{1/p} |E_2|^{1/q} |E_3|^{1/r'}\right)^{\theta} \left(|E_1|^1 |E_2|^1 |E_3|^{-1}\right)^{1-\theta} \\ &= |E_1|^{1/p_0} |E_2|^{1/q_0} |E_3|^{\theta/r' - (1-\theta)} \\ &= |E_1|^{1/p_0} |E_2|^{1/q_0} |E_3|^{1/r_0'}\end{aligned}$$

and this shows that the operator Λ satisfies the r.w.t. estimate at the point (p_0, q_0, r_0). $\square$

Collecting the results of this chapter together gives

Theorem 4.12. *Assume that* Λ *is a bilinear operator satisfying*

$$\|\Lambda(f,g)\|_{L^r} \lesssim \|f\|_{L^p}\|g\|_{L^q}, \qquad \|\Lambda(f,g)\|_{L^{1/2,\infty}} \lesssim \|f\|_{L^1}\|g\|_{L^1}$$

for $1 < p,q,r < \infty$ *satisfying the relation* $1/p + 1/q = 1/r$*. Then* Λ *is bounded for all* $r > 1/2$ *and* $1 < p,q < \infty$ *satisfying the relation* $1/p + 1/q = 1/r$*, i.e.*

$$\|\Lambda(f,g)\|_{L^r} \lesssim \|f\|_{L^p}\|g\|_{L^q}.$$

Corollary 4.13. *Assume that* Λ *is either a bilinear shift or a bilinear paraproduct. Then for* $1 < p,q < \infty$ *and* $r > 1/2$ *satisfying the relation* $1/p + 1/q = 1/r$ *it holds that*

$$\|\Lambda(f,g)\|_{L^r} \lesssim \|f\|_{L^p}\|g\|_{L^q}.$$

Remark 4.14. On how to include the cases $p,q = \infty$ in corollary 4.13, see case 3 in the proof of theorem 5.2.

5 Boundedness of the bilinear Calderón-Zygmund operator

5.1 The general setting

Assume that we have been given a set Ω equipped with a probability measure $\mathbb{P}$ and a way of assigning to each $\omega \in \Omega$ a dyadic grid $\mathcal{D}_\omega$. Expectation of a measurable function $f : \Omega \longrightarrow \mathbb{C}$ is defined as

$$\mathbb{E}f = \int_\Omega f \, d\mathbb{P}.$$

Assume that we have been given a bilinear operator Λ a priori defined on the space $L_c^\infty \times L_c^\infty$ so that

$$\langle \Lambda(f_1, f_2), f_3 \rangle$$

is defined for all $f_i \in L_c^\infty$, $i = 1, 2, 3$.

Assume that for all $f_i \in L_c^\infty$, $i = 1, 2, 3$, we have the identity

$$\langle \Lambda(f_1, f_2), f_3 \rangle = C_\Lambda \mathbb{E} \sum_{k=(k_1,k_2,k_3)\in\mathbb{N}^3} \alpha_k \sum_u \langle U_{k,u,\mathcal{D}_\omega}(f_1, f_2), f_3 \rangle.$$

Here $C_\Lambda > 0$, the sequence $(\alpha_k)_k$ is defined by

$$\alpha_k = 2^{-\alpha \max(k_i)},$$

the u summation is finite, and the operator $U_{k,u,\mathcal{D}_\omega}$ is either a shift or a paraproduct defined on the grid $\mathcal{D}_\omega$ depending on the complexity k as:

1. $U_{k,u,\mathcal{D}_\omega} = S^{k_1,k_2,k_3}(f_1, f_2)$, if $(k_1, k_2, k_3) \neq 0$,
2. If $(k_1, k_2, k_3) = 0$, then

$$U_{k,u,\mathcal{D}_\omega} = \Pi_\alpha(f_1, f_2) \text{ or } U_{k,u,\mathcal{D}_\omega} = S^{0,0,0}(f_1, f_2),$$

   where $||(\alpha_Q)_{Q\in\mathcal{D}_\omega}||_{BMO_{\mathcal{D}_\omega}(2)} \leq 1$.

In a situation like this, the boundedness of the model operators, of shifts and paraproducts, can be extended to the operator Λ.

Theorem 5.2. *Assume that* $1/2 < r < \infty$ *and* $1 < p, q \leq \infty$ *satisfy the relation* $1/p + 1/q = 1/r$. *Then*

$$\|\Lambda(f_1, f_2)\|_{L^r} \lesssim \|f_1\|_{L^p} \|f_2\|_{L^q}.$$

Proof. Since L_c^∞ is dense in L^p for $p > 0$, for the cases $p, q < \infty$ it is enough to prove that for all $f_i \in L_c^\infty$, $i = 1, 2, 3$, it holds that

$$\|\Lambda(f_1, f_2)\|_{L^r} \lesssim \|f_1\|_{L^p} \|f_2\|_{L^q},$$

since then we can extend the operator Λ boundedly to the whole space $\Lambda : L^p \times L^q \longrightarrow L^r$.

Case 1: Assume that $p, q, r > 1$. For an argument by duality let f_3 be such that $\|f_3\|_{L^{r'}} \leq 1$. By assumption on Λ and the ω-independent boundedness of shifts and paraproducts in the Banach range we can estimate

$$\begin{aligned}
|\langle \Lambda(f_1, f_2), f_3\rangle| &= |C_\Lambda \mathbb{E} \sum_{k=(k_1,k_2,k_3)\in\mathbb{N}^3} \alpha_k \sum_u \langle U_{k,u,\mathcal{D}_\omega}(f_1, f_2), f_3\rangle| \\
&\leq C_\Lambda \mathbb{E} \sum_{k=(k_1,k_2,k_3)\in\mathbb{N}^3} \alpha_k \sum_u |\langle U_{k,u,\mathcal{D}_\omega}(f_1, f_2), f_3\rangle| \\
&\lesssim C_\Lambda \mathbb{E} \sum_{k=(k_1,k_2,k_3)\in\mathbb{N}^3} \alpha_k \|f_1\|_{L^p} \|f_2\|_{L^q} \|f_3\|_{L^{r'}} \\
&\lesssim C_\Lambda \|f_1\|_{L^p} \|f_2\|_{L^q} \|f_3\|_{L^{r'}} \mathbb{E} \sum_{k=(k_1,k_2,k_3)\in\mathbb{N}^3} \alpha_k \\
&\lesssim \|f_1\|_{L^p} \|f_2\|_{L^q}.
\end{aligned}$$

Case 2: Assume that $1/2 < r \leq 1$ and $1 < p, q < \infty$. By the interpolation technique of chapter 3. it's enough to prove the weak end-point estimate

$$\|\Lambda(f_1, f_2)\|_{L^{1/2,\infty}} \lesssim \|f_1\|_{L^1} \|f_2\|_{L^1}.$$

For this, it is enough to check the major subset condition 2. of lemma 3.2.

We return to a detail in the proofs of the weak end-point estimates of shifts and paraproducts, theorems 3.5 and 3.7. At their beginning we chose a set

$$\Omega = \{\mathcal{M}_\Delta f > C|E|^{-1}\} \cup \{\mathcal{M}_\Delta g > C|E|^{-1}\}$$

and then proceeded to define the major subset E' of E as

$$E' = E \setminus \Omega.$$

We could just as well have chosen

$$\Omega = \{\mathcal{M} f > C|E|^{-1}\} \cup \{\mathcal{M} g > C|E|^{-1}\}$$

and have ran the proof through with $\mathcal{M}$ instead of $\mathcal{M}_\mathcal{D}$. Then

$$E' = E \setminus \Big(\{\mathcal{M} f > C'|E|^{-1}\} \cup \{\mathcal{M} g > C'|E|^{-1}\}\Big)$$

is a major subset that will work independent of the dyadic grid we are in, that will work independent of ω.

Now consider $0 < |E| < \infty$ and choose a major subset E' of E independent of ω. Then for f_3 such that $|f_3| \leq 1_{E'}$, especially $f_3 \in L^\infty_c$, by the reinstalled conclusions of theorems 3.5 and 3.7

$$\langle \Lambda(f_1, f_2), f_3\rangle = C_\Lambda \mathbb{E} \sum_{k=(k_1,k_2,k_3)\in\mathbb{N}^3} \alpha_k \sum_u \langle U_{k,u,\mathcal{D}_\omega}(f_1, f_2), f_3\rangle$$

$$
\begin{aligned}
&\lesssim C_\Lambda \mathbb{E} \sum_{k=(k_1,k_2,k_3)\in\mathbb{N}^3} \alpha_k \sum_u (1+\max(k_i))^2 \|f_1\|_{L^1}\|f_2\|_{L^1}|E'|^{-1} \\
&\lesssim C_\Lambda \|f_1\|_{L^1}\|f_2\|_{L^1}|E'|^{-1} \mathbb{E} \sum_{k=(k_1,k_2,k_3)\in\mathbb{N}^3} \alpha_k (1+\max(k_i))^2 \\
&\lesssim C_\Lambda \|f_1\|_{L^1}\|f_2\|_{L^1}|E'|^{-1}.
\end{aligned}
$$

Case 3: Assume that $p=\infty$. Then $r=q>1$. We extend the operator Λ to $L^\infty \times L^q$ via its first adjoint through the identity

$$\langle \Lambda(f_1,f_2), f_3\rangle = \langle \Lambda^{1*}(f_3,f_2), f_1\rangle.$$

By Fubini's theorem

$$\langle \Lambda^{1*}(f_3,f_2), f_1\rangle = C_\Lambda \mathbb{E} \sum_{k=(k_1,k_2,k_3)\in\mathbb{N}^3} \alpha_k \sum_u \langle U^{1*}_{k,u,\mathcal{D}_\omega}(f_3,f_2), f_1\rangle.$$

Since adjoints of shifts are shifts and adjoints of paraproducts are paraproducts we have boundedness for operators $U^{1*}_{k,u,\mathcal{D}_\omega} : L^{q'} \times L^q \longrightarrow L^1$. Then

$$
\begin{aligned}
\|\Lambda(f_1,f_2)\|_{L^r} &= \sup_{\|f_3\|_{L^{q'}}\le 1} \langle \Lambda(f_1,f_2), f_3\rangle = \sup_{\|f_3\|_{L^{q'}}\le 1} \langle \Lambda^{1*}(f_3,f_2), f_1\rangle \\
&= \sup_{\|f_3\|_{L^{q'}}\le 1} C_\Lambda \mathbb{E} \sum_{k=(k_1,k_2,k_3)\in\mathbb{N}^3} \alpha_k \sum_u \langle U^{1*}_{k,u,\mathcal{D}_\omega}(f_3,f_2), f_1\rangle \\
&\lesssim \sup_{\|f_3\|_{L^{q'}}\le 1} C_\Lambda \mathbb{E} \sum_{k=(k_1,k_2,k_3)\in\mathbb{N}^3} \alpha_k \|f_1\|_{L^\infty}\|f_2\|_{L^q}\|f_3\|_{L^{q'}} \\
&\lesssim \|f_1\|_{L^\infty}\|f_2\|_{L^q}.
\end{aligned}
$$

If $q=\infty$, then we extend the operator Λ via its second adjoint

$$\langle \Lambda(f_1,f_2), f_3\rangle = \langle \Lambda^{2*}(f_1,f_3), f_2\rangle$$

and carry through as in the case $p=\infty$. $\square$

5.3 Definition of a bilinear Calderón-Zygmund operator

Definition 5.4. *A mapping $K\colon \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n \setminus \Delta \longrightarrow \mathbb{C}$, where*

$$\Delta = \{(x,y,z)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n : x=y=z\},$$

is a standard Calderón-Zygmund α-kernel if for some $\alpha\in(0,1]$ and $C_K<\infty$ it holds that

$$|K(x,y,z)| \le \frac{C_K}{(|x-y|+|x-z|)^{2n}},$$

$$|K(x,y,z)-K(x',y,z)| \leq C_k \frac{|x-x'|^\alpha}{(|x-y|+|x-z|)^{2n+\alpha}}$$

whenever $|x-x'| \leq \max(|x-y|,|x-z|)/2$,

$$|K(x,y,z)-K(x,y',z)| \leq C_k \frac{|y-y'|^\alpha}{(|x-y|+|x-z|)^{2n+\alpha}}$$

whenever $|y-y'| \leq \max(|x-y|,|x-z|)/2$,

$$|K(x,y,z)-K(x,y,z')| \leq C_k \frac{|z-z'|^\alpha}{(|x-y|+|x-z|)^{2n+\alpha}}$$

whenever $|z-z'| \leq \max(|x-y|,|x-z|)/2$.

The best constant C_K *is denoted by* $\|K\|_{CZ_\alpha}$.

Definition 5.5. *Let* $\mathcal{F}$ *denote the set of all finite linear combinations of characteristic functions of cubes in* $\mathbb{R}^n$. *Assume that we have a bilinear map* T *and its adjoints* T^{1*} *and* T^{2*} *defined on* $\mathcal{F}$, *i.e. for* $f_i \in \mathcal{F}$, $i = 1,2,3$,

$$\langle T(f_1,f_2),f_3\rangle = \langle T^{1*}(f_3,f_2),f_1\rangle = \langle T^{2*}(f_1,f_3),f_2\rangle,$$

and that they map

$$T, T^{1*}, T^{2*} : \mathcal{F} \times \mathcal{F} \longrightarrow L^1_{loc}.$$

An operator $T : \mathcal{F} \times \mathcal{F} \longrightarrow L^1_{loc}$ *is a bilinear Calderón-Zygmund operator if exists a standard Calderón-Zygmund* α*-kernel* K *so that* T *has a representation in the following sense: For all* $f_i \in \mathcal{F}$, $i,j = 1,2,3$, *on the condition that for some* $i \neq j$

$$\operatorname{spt} f_i \cap \operatorname{spt} f_j = \emptyset,$$

it holds that

$$\langle T(f_1,f_2),f_3\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y,z) f_1(y) f_2(z) f_3(x)\, dydzdx.$$

Remark 5.6. Heuristically, if T is a linear Calderón-Zygmund operator on $\mathbb{R}^{2n}$, then $T'(f_1,f_2)(x) = T(f_1 \otimes f_2)(x,x)$ is a linear Calderón-Zygmund on $\mathbb{R}^n$.

Definition 5.7. *A bilinear Calderón-Zygmund operator* T *is said to satisfy local T1 conditions if exists a constant C>0 so that for all cubes* I *it holds that*

$$|\langle T(1_I,1_I),a_I\rangle| + |\langle T^{1*}(1_I,1_I),a_I\rangle| + |\langle T^{2*}(1_I,1_I),a_I\rangle| \leq C|I|,$$

for all $a_I \in \mathcal{F}$ *such that*

$$\operatorname{spt}(a_I) \subset I, \qquad |a_I| \leq 1.$$

The best constant C *is denoted by* C_{T1}.

5.8 Statement of the representation theorem for bilinear Calderón-Zygmund operators

We are almost done, it only remains to pull everything together.

Theorem 5.9. *Let T be a bilinear Calderón-Zygmund operator associated to a kernel of magnitude α and assume that T satisfies the local T1 conditions. Then T can be extended so that for all $f_i \in L^\infty_c$, $i = 1, 2, 3$, $\langle T(f_1, f_2), f_3\rangle$ is defined and we have representation as described in the beginning of this chapter:*

$$\langle T(f_1, f_2), f_3\rangle = C_T \mathbb{E} \sum_{k=(k_1,k_2,k_3)\in\mathbb{N}^3} \alpha_k \sum_u \langle U_{k,u,\mathcal{D}_\omega}(f_1, f_2), f_3\rangle.$$

More the constant C_T satisfies

$$C_T \leq \|K\|_{CZ_\alpha} + C_{T1}.$$

Proof. See [3]. □

Corollary 5.10. *For exponents $p, q > 1$ and $r > 1/2$ satisfying the relation $1/p + 1/q = 1/r$, a bilinear Calderón-Zygmund operator T is bounded as a mapping*

$$T : L^p \times L^q \longrightarrow L^r.$$

Proof. Follows immediately from theorems 5.9 and 5.2. □

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